S. Griffeth 9/6/06

Rational Cherednik algebras

k a field, V a fin. dim'l k-vector space

$W \subseteq GL(V)$ a finite subgroup

For each $w \in W$ fix a skew-symmetric bilinear form $\langle \cdot, \cdot \rangle_w$ on V

The Drinfeld Hecke algebra $\mathbb{H}$ for this data

has generators $x \in V$, t_w for $w \in W$

and relations

in particular, $t_w x t_w^{-1} = w(x)$

$$t_1 = 1 \qquad t_v t_w = t_{vw} \qquad t_w x = w(x) t_w \qquad \forall v, w \in W,\ x \in V$$

$$\text{and} \quad [x,y] := xy - yx = \sum_{w \in W} \langle x,y \rangle_w t_w$$

twisted gp a'g.

$$\mathbb{H} = TV \otimes_{\mathbb{C}} \mathbb{C}W \Big/ \left\langle xy - yx = \sum_{w \in W} \langle x,y \rangle_w t_w \right\rangle$$

Example: If $\langle \cdot, \cdot \rangle_w = 0 \ \forall w \in W$,

then $\mathbb{H} \cong S(V) \otimes_k kW$

with multiplication $(f \otimes t_w) \cdot (g \otimes t_v) = f \cdot w(g) \otimes t_{wv}$

(= twisted group algebra)

Example: The (graded) degenerate affine Hecke algebra is a special case.

Example: If W is a complex ref'n group,

$\mathfrak{h}$ its ref'n rep'n

$V = \mathfrak{h} \oplus \mathfrak{h}^*$ w/ diagonal W action

and set
$$\langle x,y \rangle_w = \begin{cases} 0 & \text{unless } w = 1 \text{ or } w = s \in T = \text{all ref'n} \\ k \langle x,y \rangle & \\ c_s \langle x - sx, y - s^{-1}y \rangle \dfrac{1}{1 - \det_{\mathfrak{h}^*} s} & \text{if } x \in \mathfrak{h}^*,\ y \in \mathfrak{h} \end{cases}$$

$= c_s \langle \alpha_s, y \rangle \langle x, \alpha_s^\vee \rangle$

where we've fixed scalars $k \in \mathbb{C}$

$c_s \in \mathbb{C} \ \forall s \in T$

with $c_{wsw^{-1}} = c_s \quad \forall w \in W,\ s \in T$

is called the rational Cherednik algebra

PBW Thm for $\mathbb{H}$

Say the PBW theorem holds for $\mathbb{H}$ if for any basis $x_1, \dots, x_n$ of V, the set

$$\{ x_{i_1} x_{i_2} \cdots x_{i_p} t_w \mid 1 \le i_1 \le \dots \le i_p \le n \}$$

is a k-basis for $\mathbb{H}$.

THEOREM:

PBW holds for $\mathbb{H}$ $\iff$

(a) $\langle vx, vy \rangle_{vwv^{-1}} = \langle x, y \rangle_w \quad \forall\, x, y \in V \quad v, w \in W$

(b) $\langle x, y \rangle_w (wz - z) + \langle y, z \rangle_w (wx - x) + \langle z, x \rangle_w (wy - y) = 0.$

proof: ($\Longrightarrow$):

$$[vx, vy] = t_v [x, y] t_v^{-1} = t_v \Big(\sum_{w \in W} \langle x, y \rangle_w t_w \Big) t_v^{-1}$$

$$\sum_{w \in W} \langle vx, vy \rangle_w t_w = [vx,vy] \qquad \sum_{w \in W} \langle x, y \rangle_w t_{vwv^{-1}} = t_v \Big(\sum_{w \in W} \langle x, y \rangle_w t_w \Big) t_v^{-1}$$

$$\Downarrow \text{PBW}$$

$\langle vx, vy \rangle_{vwv^{-1}} = \langle x, y \rangle_w$ proving (a).

KEY FACT: $t_w z - z t_w = (w(z) - z) t_w$

For all $x, y, z \in V$

$$0 = [[x,y],z] + [[y,z],x] + [[z,x],y]$$

$$= \Big[\sum_{w \in W} \langle x, y \rangle_w t_w, z \Big] + \Big[\sum_{w \in W} \langle y, z \rangle_w t_w, x \Big] + \Big[\sum_{w \in W} \langle z, x \rangle_w t_w, y \Big]$$

$$= \sum_{w \in W} \Big(\langle x, y \rangle_w (wz - z) + \langle y, z \rangle_w (wx - x) + \langle z, x \rangle_w (wy - y) \Big) t_w$$

$$\Downarrow \text{PBW}$$

(b) holds also.

($\Longleftarrow$): 1st show the given set spans;

easy, as the relations let you move t_w's to the right and put the x_i's in order using the commutators $[x_i, x_j] \in$ k-span of t_w's.

For lin. independence ....

$\omega^N = 1$

Let M be the k-vector space with k-basis (imagine it's the left-regular rep'n of $\mathbb{H}$ if PBW held

$$\{ x_{i_1} \cdots x_{i_p} t_w \mid 1 \le i_1 \le \ldots \le i_p \le n,\ w \in W \} \quad p \in \mathbb{Z}_{\ge 0}$$

Define operators l_x and l_w on M inductively:

BASE CASE
$$\begin{cases} l_x \cdot t_w = x t_w & \forall x \in V,\ v, w \in W \\ l_v \cdot t_w = t_{vw} \end{cases}$$

INDUCTIVE:
$$l_v \cdot x_{i_1} \cdots x_{i_p} t_w = l_{v x_{i_1}} \cdot (l_v \cdot x_{i_2} \cdots x_{i_p} t_w)$$

$$l_{x_i} x_{i_1} \cdots x_{i_p} t_w = \begin{cases} x_i x_{i_1} \cdots x_{i_p} t_w & \text{if } i \le i_1 \\ l_{x_{i_1}} \cdot l_{x_i} \cdot x_{i_2} \cdots x_{i_p} t_w + \sum_{v \in W} \langle x_i, x_{i_1} \rangle l_v x_{i_2} \cdots x_{i_p} t_w & \text{if } i > i_1 \end{cases}$$

For the proof that this gives a rep'n of $\mathbb{H}$, and shows lin. indep, see Stephen's notes on-line. ∎

Assuming $|W| \in k^\times$

<u>COR</u>: PBW holds for $\mathbb{H}$ iff

(i) $\langle vx, vy \rangle_{vwv^{-1}} = \langle x, y \rangle_w$

(ii) $\langle \cdot, \cdot \rangle_w = 0$ unless $w = 1$ or $\operatorname{codim}(V^w) = 2$

(iii) $V^w \subseteq \operatorname{Rad}\langle \cdot, \cdot \rangle_w$ for $w \neq 1$.

<u>proof</u>: $(\Rightarrow)$: For any $w \in W$, $V = (w-1)V \oplus V^w$ because $|W| \in k^\times$

Only this implication needs $|W| \in k^\times$

If $x, y \in V^w$ and $w \neq 1$, then $\langle x, y \rangle_w = 0$:

Choose $z \in V$ with $(wz - z) \neq 0$ and

(b) $\Rightarrow \langle x, y \rangle_w (wz - z) = 0 \Rightarrow \langle x, y \rangle_w = 0$.

If $x \in V^w$, $y \in (w-1)V$ then
$$\begin{aligned} \langle x, y \rangle_w = \langle x, wz - z \rangle_w &= \langle x, wz \rangle_w - \langle x, z \rangle_w \\ &= \langle w^{-1}x, z \rangle_w - \langle x, z \rangle_w \\ &= \langle x, z \rangle_w - \langle x, z \rangle_w \\ &= 0. \end{aligned}$$

So (iii) holds.

Suppose $w \neq 1$ and $\operatorname{codim} V^w \neq 2$.

Then by (iii), $V^w \subseteq \operatorname{Rad}\langle \cdot, \cdot \rangle_w$

Assume for contradiction that $\langle \cdot, \cdot \rangle_w \neq 0$.

Pick $x, y \in V$ with $\langle x, y \rangle_w = 1$.

By (b), $(wz - z) + \langle y, z \rangle_w (wx - x) + \langle z, x \rangle_w (wy - y) = 0$

$\Rightarrow wz - z \in \operatorname{Span}\{wx - x, wy - y\} \ \forall z \in V$

Hence $\dim (w-1)V \leq 2$

so $\overset{\shortparallel}{\operatorname{codim}} V^w \leq 2$.

Since $V^w \subseteq \operatorname{Rad}\langle \cdot, \cdot \rangle_w$ and $\langle \cdot, \cdot \rangle_w$ is skew-symm.,

we must have $\operatorname{codim} V^w = 2$.

$(\Leftarrow)$: We need to show (b).

Fix $\langle \cdot, \cdot \rangle_w \neq 0$.

If no 2 of x, y, z are lin. indep. mod $\operatorname{Rad}\langle \cdot, \cdot \rangle_w$,

then (b) holds (if $x = ay + r$ with $r \in \operatorname{Rad}$

then $\langle x, y \rangle_w = a \underbrace{\langle y, y \rangle_w}_{0} + \underbrace{\langle r, y \rangle_w}_{0}$)

So assume x, y are lin. indep. mod $\operatorname{Rad}\langle \cdot, \cdot \rangle$

and we'll prove (b) holds $\forall z \in V$.

If $w = 1$, (b) holds trivially.

If $w \neq 1$, then (ii) says $\operatorname{codim} V^w = 2$ and (iii) says $V^w = \operatorname{Rad}\langle \cdot, \cdot \rangle_w$

Thus x, y are lin. indep. mod $\ker(w-1)$

$\cancel{\Rightarrow wx - x, wy - y \text{ span } (w-1)V}$

so $\forall z \in V$, $\cancel{wz - z = a(wx - x) + b(wy - y)}$

$z = ax + by \mod \operatorname{Rad}\langle \cdot, \cdot \rangle_w$ for some $a, b \in k$

$\Rightarrow \langle y, z \rangle_w = a \langle y, x \rangle_w \neq 0$, so $a = \dfrac{\langle y, z \rangle_w}{\langle y, x \rangle_w}$

and $\langle z, x \rangle_w = b \langle y, x \rangle_w \neq 0$ $\quad b = \dfrac{\langle y, x \rangle_w}{\langle z, x \rangle_w}$

So $z = \dfrac{\langle y, z \rangle_w}{\langle y, x \rangle_w} x + \dfrac{\langle z, x \rangle_w}{\langle y, x \rangle_w} y \mod \operatorname{Rad}\langle \cdot, \cdot \rangle = V^w = \ker(w-1)$

Apply $w - 1$ to both sides...

$$wz - z = \frac{\langle y,z\rangle_w}{\langle y,x\rangle_w}(wx - x) + \frac{\langle z,x\rangle_w}{\langle y,x\rangle_w}(wy - y)$$

Rearranging gives (b). ■

Let $W \subset GL(\mathfrak{h})$ be a reflection group. Fix $\kappa \in k$ and $c_s \in k$ for $s \in T$ with $c_{wsw^{-1}} = c_s$

Let $V = \mathfrak{h}^* \oplus \mathfrak{h}$

Define forms $\langle \cdot,\cdot\rangle_w$ on V by:

(a) $\langle \cdot,\cdot\rangle_w$ is skew-symmetric

(b) $\underset{\kappa\langle\cdot,\cdot\rangle}{\overset{\shortparallel}{\langle \cdot,\cdot\rangle_1}}$ is κ times the canonical pairing $\mathfrak{h}^* \otimes \mathfrak{h} \to k$

$(\text{so } \langle \mathfrak{h},\mathfrak{h}\rangle = \langle \mathfrak{h}^*,\mathfrak{h}^*\rangle = 0)$

(c) If $s \in T$ is a refl'n, define $\alpha_s, \alpha_s^\vee$ by

$$s(x) = x - \langle x, \alpha_s^\vee\rangle \alpha_s \quad \forall x \in \mathfrak{h}^*$$

i.e. α_s spans $\mathrm{im}(1-s)$ and $x - sx = (?)\,\alpha_s$ (linear in x, call it $\langle x, \alpha_s^\vee\rangle$)

then set $\langle x,y\rangle_s := c_s \langle \alpha_s, y\rangle \langle x, \alpha_s^\vee\rangle$

(d) $\langle \cdot,\cdot\rangle_w = 0 \quad \forall\, w \notin T,\ w \neq 1$

Then (i), (ii), (iii) holds so PBW holds for $\mathbb{H}_c$.

<u>Rephrasing</u>: $\underset{k\text{-vectorspaces}}{\mathbb{H}_c \cong} \underset{\text{basis: } x_1,\dots,x_n}{S(\mathfrak{h}^*)} \otimes \mathbb{C}W \otimes \underset{\text{basis: } y_1,\dots,y_n}{S(\mathfrak{h})}$

<u>COR</u>: $\mathbb{H}_c$ acts on $S(\mathfrak{h}^*)$ by

$$x.f = xf$$
$$y.f = \kappa\,\partial_y f - \sum_{s \in T} c_s \langle \alpha_s, y\rangle \frac{f - sf}{\alpha_s}$$
$$t_w.f = w(f)$$

$\forall x \in \mathfrak{h}^*, f \in S(\mathfrak{h}^*), y \in \mathfrak{h}, w \in W$

In particular, all these y operators commute w/ each other!

proof: Define $M_c(\mathbb{1}) := \mathbb{H}_c \otimes_{S(h)\otimes\mathbb{C}W} \mathbb{1}$

where $\mathbb{1}$ is the $S(h)\otimes\mathbb{C}W$-module

w/ basis elt 1 and action $t_w \cdot 1 = 1$

$y \cdot 1 = 0 \ \forall w \in W, y \in$

By PBW, $M_c(\mathbb{1}) \cong S(h^*)$

Check by induction that the $y.f$ formula holds

(the $x.f$, $w.f$ formulae are clear) ■

10/10/06 $\quad V := h^* \oplus h$

$\cancel{\mathbb{H}_c = TV \otimes_{\mathbb{C}} \mathbb{C}W}$, $W \subset GL(h)$ a complex refl'n gp.

Fix $k \in \mathbb{C}$, $c_s \in \mathbb{C}$ $\forall s \in T$ fixed so that $c_s = c_{wsw^{-1}}$ $\forall w \in W$ $s \in T$

$$\mathbb{H}_c := TV \otimes_{\mathbb{C}} \mathbb{C}W \Big/ \left(yx - xy = -\sum_{s\in T} c_s \langle \alpha_s, y\rangle \langle x, \alpha_s^\vee \rangle t_s + k\langle x, y\rangle \quad x_1x_2 = x_2x_1 \quad y_1y_2 = y_2y_1 \right)$$

$sx = x - \langle x, \alpha_s^\vee \rangle \alpha_s$

PROP: $\forall y \in h$ and $f \in S(h^*)$

$$(*) \qquad [y, f] = k\partial_y f - \sum_{s\in T} c_s \langle \alpha_s, y\rangle \frac{f - sf}{\alpha_s} t_s$$

DEF'N:
$\partial_y(x) = \langle x, y\rangle$ $\forall x \in h^*$
and
$\partial_y(fg) = (\partial_y f)g + f \cdot \partial_y g$

proof: Induct on $\deg f$. If $f = x \in h^*$,

$$[y, x] = k\langle x, y\rangle - \sum_{s\in T} c_s \langle \alpha_s, y\rangle \langle x, \alpha_s^\vee\rangle t_s$$

$$= k\partial_y(x) - \sum_{s\in T} c_s \langle \alpha_s, y\rangle \frac{x - sx}{\alpha_s} t_s .$$

Suppose $(*)$ holds for f and $g \in S(h^*)$.

Then...

$$[y,fg] = [y,f]g + f[y,g]$$
$$(= yfg - fgy) \qquad (= (yf - \cancel{fy})g + \cancel{f}(yg - gy))$$

$$= \left(\kappa\partial_y f - \sum_{s\in T} c_s\langle\alpha_s,y\rangle \frac{f-sf}{\alpha_s} t_s\right) g$$
$$+ f\left(\kappa\partial_y g - \sum_{s\in T} c_s\langle\alpha_s,y\rangle \frac{g-sg}{\alpha_s} t_s\right)$$

$$= \kappa\partial_y(fg) - \sum_{s\in T} c_s\langle\alpha_s,y\rangle\left(\frac{f-sf}{\alpha_s}(sg)t_s + f\frac{g-sg}{\alpha_s}t_s\right)$$

$$= \kappa\partial_y(fg) - \sum_{s\in T} c_s\langle\alpha_s,y\rangle \frac{fg - s(fg)}{\alpha_s} t_s \qquad \blacksquare$$

COR: The action $y\in\mathfrak{h}$ on $S(\mathfrak{h}^*) = M(\mathrm{triv})$
$$:= \mathrm{Ind}^{\mathbb{H}_c}_{S(\mathfrak{h})\otimes\mathbb{C}W}\mathrm{triv}$$
$$= \mathbb{H}_c \otimes_{S(\mathfrak{h})\otimes\mathbb{C}W}\mathrm{triv}$$

is $y.f = \kappa\partial_y f - \sum_{s\in T} c_s\langle\alpha_s,y\rangle\frac{f-sf}{\alpha_s}$

proof: By def'n,
$$y.f = y.(f\otimes 1) \qquad (\text{i.e. } \mathrm{triv} = \mathbb{C}1)$$
$$= yf\otimes 1$$
$$= ([y,f]+fy)\otimes 1$$
$$= [y,f]\otimes 1 + f\otimes \underbrace{y.1}_{0}$$
$$= \left(\kappa\partial_y f - \sum_{s\in T} c_s\langle\alpha_s,y\rangle\frac{f-sf}{\alpha_s}t_s\right)\otimes 1$$
$$= \left(\kappa\partial_y f - \sum_{s\in T} c_s\langle\alpha_s,y\rangle\frac{f-sf}{\alpha_s}\right)\otimes 1. \qquad \blacksquare$$