Semisimple Algebras (following Webb Ch. 2) 6/15/17

Maschke's Thm: Let (ρ, V) be a rep'n of the finite gp G over a field F where $|G|$ is invertible. If W is an invariant subsp. of V, $\exists$ inv. ssp W_1 s.t $V = W \oplus W_1$ (as rep'ns).

Pf: $\langle u, v\rangle_\rho = \frac{1}{|G|} \sum_{g \in G} \langle \rho(u), \rho(v)\rangle$ is a G-inv inner prod. (explain), so ρ is called unitary. Therefore, Let $W_1 = \{v \in V : \langle v, w\rangle_\rho = 0 \ \forall w \in W\}$. Clearly $V = W \oplus W_1$ as vsp's, and if $v \in W_1$, $w \in W$, $\langle \rho(v), w\rangle_\rho = \langle v, \rho^{-1}(w)\rangle_\rho = 0$.

Cor: ρ as above is semisimple (completely reducible).

Eg (not semisimple): $G = \cancel{C_p}\ \mathbb{Z}/p\mathbb{Z}$, $F = \mathbb{F}_p$, $\rho(g^k) = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$.

Schur's Lemma (part 1): Let A be a ring with 1, and let S_1, S_2 be simple A-modules. Then $\mathrm{Hom}_A(S_1, S_2) = 0$ unless $S_1 \cong S_2$, in which case $\mathrm{End}_A(S_1)$ is a division ring.

Pf: Let $\theta: S_1 \to S_2$. $\mathrm{Im}(\theta)$ and $\ker(\theta)$ are submods of S_2, S_1, resp. Thus θ is either an isom or 0. If θ is an isom, it has an inverse, so $\mathrm{End}_A(S_i)$ is a division ring.

Schur's Lemma (part 2): If A is a f.d. alg/k, ← alg. closed, then $\mathrm{End}_A(S_1) \cong k$.

Pf: Let $\theta \in \mathrm{End}_A(S_1)$, and let λ be an e-value of θ. $\theta - \lambda$ is therefore singular, and thus 0, so $\theta = \lambda I$.

We call k a splitting field for A.

An algebra is called "semi-simple" if every A-module is semisimple $\Leftrightarrow$ ${}_AA$ is semisimple ← reg rep'n

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Thm (Artin-Wedderburn): Let A be a fin. dim. semisimple alg/k: field. If ${}_AA \cong S_1^{n_1} \oplus \dots \oplus S_r^{n_r}$, then $A \cong M_{n_1}(D_1) \oplus \dots \oplus M_{n_r}(D_r)$ where $D_i = \mathrm{End}_A(S_i)^{op}$. (If k: alg closed, $D_i = k$ by Schur's Lemma part 2).

(First: For any ring A with 1, $\mathrm{End}_A({}_AA) \cong A^{op}$.) ← ring isom.

Pf: Any endom. of $U = U_1 \oplus \dots \oplus U_m$ consists of homoms. $U_i \to U_j$ $\forall i,j$. We know that $\mathrm{Hom}_A(S_i, S_j) = 0$, $i \neq j$, and that $\mathrm{Hom}_A(S_i, S_i) = D_i^{op}$, so $\mathrm{End}_A(S_i^{n_i}) \cong M_{n_i}(D_i^{op}) \cong M_{n_i}(D_i)^{op}$. Therefore,

$$A \cong \mathrm{End}_A({}_AA)^{op} \cong \mathrm{End}_A(S_1^{n_1})^{op} \oplus \mathrm{End}_A(S_r^{n_r})^{op} \cong M_{n_1}(D_1) \oplus \dots \oplus M_{n_r}(D_r).$$

Cor: Let A be a f.d. semisimple alg/k: field. If ${}_AA = S_1^{n_1} \oplus \cdots \oplus S_r^{n_r}$, $S_i \not\cong S_j$, then every simple A-module is isom to some S_i.

When k is alg. closed, $n_i = \dim_k S_i$ and $\dim_k A = n_1^2 + \cdots + n_r^2$.

Pf: Every simple module is a homomorphic image of ${}_AA$, thus is isom to some S_i.

Now, if k: alg closed, $A = M_{n_1}(k) \oplus \cdots M_{n_r}(k)$, so has dim $n_1^2 + \cdots + n_r^2$. In particular, we have this for $A = M_{n_i}(k)$, so $\dim(S_i^{n_i}) = n_i^2$, so $\dim(S_i) = n_i$.