representation theory
seminar

# Introduction to Quiver Algebras

Informally, a quiver is a labeled directed graph with loops and multiple edges allowed.

Ex|| $\bullet \rightleftarrows \bullet \qquad \bullet\circlearrowleft \qquad \begin{matrix}\bullet\\ \bullet\end{matrix}\!\!\rightrightarrows \bullet \to \bullet$ are all quivers

Formally,

**Definition:** A <u>quiver</u> is a quadruple $(Q_0, Q_1, s, t)$ where $Q_0 := \{\text{vertices}\}$, $Q_1 := \{\text{arrows}\}$, and $s, t: Q_1 \to Q_0$ are maps identifying the source and target, respectively, of arrows in $Q_1$.

Ex|| $\overset{\bullet}{1} \xrightarrow{\alpha} \overset{\bullet}{2} \qquad s(\alpha) = 1,\ t(\alpha) = 2$

**Definition:** $KQ$, the <u>path algebra</u> or <u>quiver algebra</u> is a $K$-vector space (algebraically closed field) with the set of all paths of length $\geq 0$ as its basis.

Ex|| $Q = \overset{1}{\bullet} \xrightarrow{\alpha} \overset{2}{\bullet}$ has $KQ = \text{span}\{\underbrace{e_1, e_2}_{\text{lazy paths (aka trivial paths)}}, \alpha\}$

The operation in $KQ$ is defined by concatenation: $\beta \cdot \alpha = \begin{cases} \beta\alpha & \text{if } t(\alpha) = s(\alpha) \\ 0 & \text{else} \end{cases}$

Ex|| $\overset{1}{\bullet} \xrightarrow{\alpha} \overset{2}{\bullet} \xrightarrow{\beta} \overset{3}{\bullet} \qquad \beta \cdot \alpha = \beta\alpha\ ,\ \beta \cdot e_2 = \beta,\ \beta \cdot e_3 = 0$, etc...

The identity element is the sum of the lazy paths. In the previous example,

$$\alpha(e_1 + e_2 + e_3) = \alpha \cdot e_1 + \alpha \cdot e_2 + \alpha \cdot e_3 = \alpha \cdot e_1 = \alpha$$

Some examples of quiver algebras:

Ex|| $1 \bullet \circlearrowleft^{\alpha}$ has $KQ = \text{span}\{e_1, \alpha, \alpha^2, \alpha^3, \dots\} \simeq K[t]$

$\alpha \mapsto t$

$e_1 \mapsto 1$

Ex|| $\alpha \circlearrowleft \overset{1}{\bullet} \circlearrowright \beta \quad KQ = \{\text{words on } \alpha, \beta \text{ with empty word} = e_1\} \simeq K\langle \underbrace{t_1, t_2}_{\text{non-commuting variables}} \rangle$

$\varepsilon_1 \mapsto 1$

$\alpha \mapsto t_1$

$\beta \mapsto t_2$

Note: KQ not always finite.

Let $I$ be a two-sided ideal of $KQ$. Then $\exists$ a sufficient condition on $KQ$ s.t. $KQ/I$ is finite.

Definition: The two-sided ideal $R_Q$ generated by the arrows of $Q$ (i.e., paths of length 1) is called the arrow ideal of $KQ$.

Definition: A two-sided ideal $I$ is admissible if $\exists\ m \geq 2$ s.t. $R_Q^m \subseteq I \subseteq R_Q^2$

Unpacking this definition – $R_Q^2$ = all paths of length 2

$R_Q^m$ = all paths of length $m$

$I$ contains paths of length $\geq 2$. $R_Q^m \subseteq I$ implies $I$ contains all paths of length $\geq m$.

But possibly some subset of paths with lengths between 2 to $m-1$.

$I \subseteq R_Q^2$ ensures $KQ/I$ is connected.

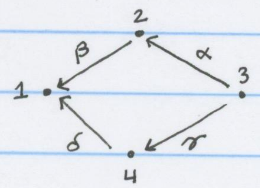
Definition: For admissible $I$, $KQ/I$ is a bound quiver algebra.

EX|| $Q = \alpha \circlearrowleft \cdot 1 \xrightarrow{\beta} \cdot 2$ $\qquad I = \langle \alpha^2\beta, \alpha^3 \rangle$ is admissible $(m=3)$

any path with length $\geq 3$ must contain $\alpha^3$ or $\alpha^2\beta$, so $R_Q^3 \subseteq I$

Clearly $I \subseteq R_Q^2$ since its basis elements have length 3

EX|| Different relations on the same quiver can give you the same bound quiver algebra.




$I_1 = \langle \alpha\beta + \gamma\delta \rangle$ $\qquad$ So $I_1 \neq I_2$, but $KQ/I_1 \simeq KQ/I_2$

$I_2 = \langle \alpha\beta - \gamma\delta \rangle$

Quivers can be used to visualize modules. For a quiver $Q$ with bound quiver algebra $A = KQ/I$, we can visualize any $A$-module, $M$, as a $K$-linear representation of $(Q, I)$

**Definition:** A K-linear representation M of Q is specified by:

(a) associating each point $a \in Q_0$ with a K-vector space $M_a$

(b) associating each arrow $\alpha \in Q_1$ with a K-linear map $\varphi_\alpha : M_{s(\alpha)} \to M_{t(\alpha)}$

We denote this as $M = (M_a, \varphi_\alpha)$.

**Definition:** M is <u>finite dimensional</u> if each $M_a$ is finite dimensional.

EX\\ $\overset{1}{\bullet} \underset{\beta}{\overset{\alpha}{\leftleftarrows}} \overset{2}{\bullet}$ has representations $K^2 \underset{\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]}{\overset{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]}{\leftleftarrows}} K$ , $K^2 \underset{\left[\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right]}{\overset{\left[\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right]}{\leftleftarrows}} K^2$

Let $Q = (Q_0, Q_1, s, t)$ be an unbound quiver. Then for each $i \in Q_0$,

**Definition:** The <u>simple representation</u> $S(i)$ has dimension 1 at vertex i and 0 at all other vertices. There is one such representation for each $i \in Q_0$.

**Definition:** The <u>projective representation</u> $P(i)$ is constructed as follows:

(a) Let $P(i)_j$ be the K-vector space with basis {all paths from i to j in Q}

(b) For each $j \xrightarrow{\alpha} \ell$ in $Q_1$, let $\varphi_\alpha : P(i)_j \to P(i)_\ell$ be the linear map defined by composing the paths from i to j with $j \xrightarrow{\alpha} \ell$.

Definition: The injective representation $I(i)$ is constructed as

(a) $I(i)_j$ is the $K$-vector space with basis {paths from $j$ to $i$ in $Q$}

(b) For $j \xrightarrow{\alpha} \ell$ in $Q$, $\varphi_\alpha : I(i)_j \to I(i)_\ell$ is the linear map defined on the basis by deleting $j \xrightarrow{\alpha} \ell$ from paths from $j$ to $i$ and sending other paths to zero.

Ex1 (easy)

$$Q = \begin{tikzcd} \overset{1}{\bullet} \arrow[r] & \overset{2}{\bullet} & \overset{3}{\bullet} \arrow[l] \arrow[d] & \overset{4}{\bullet} \arrow[l] \\ & & \bullet 5 & \end{tikzcd}$$

$$S(3) \simeq \begin{tikzcd} 0 \arrow[r] & 0 & K \arrow[l] \arrow[d] & 0 \arrow[l] \\ & & 0 & \end{tikzcd}$$

$$P(3) \simeq \begin{tikzcd} 0 \arrow[r] & K & K \arrow[l, "1"'] \arrow[d, "1"] & 0 \arrow[l] \\ & & K & \end{tikzcd}$$

$$I(3) \simeq \begin{tikzcd} 0 \arrow[r] & 0 & K \arrow[l] \arrow[d] & K \arrow[l, "1"'] \\ & & 0 & \end{tikzcd}$$

Ex2 (more complicated)

$$Q = \begin{tikzcd} \overset{1}{\bullet} \arrow[rr] \arrow[dr] & & \overset{3}{\bullet} \arrow[r] & \overset{4}{\bullet} \\ & \underset{2}{\bullet} \arrow[ur] & & \end{tikzcd}$$

$$P(1) \simeq \begin{tikzcd} K \arrow[rr, "{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}"] \arrow[dr, "1"'] & & K^2 \arrow[r, "{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}"] & K^2 \\ & K \arrow[ur, "{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}"'] & & \end{tikzcd}$$

note dimensions are $>1$ when there are multiple paths from $j$ to $i$.

$$I(4) \simeq \begin{tikzcd} K^2 \arrow[rr, "{\begin{bmatrix} 0 & 1 \end{bmatrix}}"] \arrow[dr, "{\begin{bmatrix} 1 & 0 \end{bmatrix}}"'] & & K \arrow[r, "1"] & K \\ & K \arrow[ur, "1"'] & & \end{tikzcd}$$

Let $M$ be a right $A$-module. Then:

Definition: ~~Let M be a right A-module~~. Then the radical of $M$, $\operatorname{rad} M$, is the intersection of all maximal submodules of $M$. Here, $\operatorname{rad}(KQ/I) = R_Q =$ "{arrows of $Q$}"

$\operatorname{rad}^k(KQ/I) = R_Q^k =$ "paths of length $k$ in $Q$."

Definition: ~~Let M be a right A~~ The socle of $M$, $\operatorname{soc} M$, is the submodule generated by all simple submodules of $M$. Here, $\operatorname{soc} M =$ "vectors annihilated by every paths of length 1"

$\operatorname{soc}^k M =$ " ———————— " —— $k$"

Note: all arrows in $\operatorname{Soc}(M)$ act by 0.

Definition: $\operatorname{top} M = M/\operatorname{rad} M$

~~[illegible]~~

~~[illegible]~~

~~[illegible]~~

Let $M = K \underset{[0\,0\,1]}{\overset{[0\,1\,0]}{\leftleftarrows}} K^3 \underset{\left[\begin{smallmatrix}0\\1\\0\end{smallmatrix}\right]}{\overset{\left[\begin{smallmatrix}1\\0\\0\end{smallmatrix}\right]}{\leftleftarrows}} K$ (from $Q = \overset{1}{\bullet} \leftleftarrows \overset{2}{\bullet} \leftleftarrows \overset{3}{\bullet}$)

<u>Definition:</u> $\operatorname{soc} M = N = (N_a, \psi_\alpha)$ with $N_a = M_a$ if $a$ is a sink

$N_a = \bigcap_{\alpha: a \to b} \ker(\varphi_\alpha : M_a \to M_b)$ if $a$ not a sink

$\psi_\alpha = \varphi_\alpha|_{N_a} = 0$ for every arrow $\alpha$ of the source $a$.

<u>EX!!</u> $\operatorname{Soc}(M) = K \underset{0}{\overset{0}{\leftleftarrows}} K \underset{0}{\overset{0}{\leftleftarrows}} 0$

<u>Definition:</u> $\operatorname{rad}(M) = J = (J_a, \gamma_\alpha)$ with $J_a = \sum_{\alpha: b \to a} \operatorname{Im}(\varphi_\alpha : M_b \to M_a)$

$\gamma_\alpha = \varphi_\alpha|_{J_a}$ for every arrow of source $a$

<u>EX!!</u> $\operatorname{rad} M = K \underset{0}{\overset{[0\,1]}{\leftleftarrows}} K^2 \underset{0}{\overset{0}{\leftleftarrows}} 0$

<u>Definition:</u> $\operatorname{top} M = L = (L_a, \psi_\alpha)$ with $L_a = M_a$ if $a$ is a source

$L_a = \sum_{\alpha: b \to a} \operatorname{coker}(\varphi_\alpha : M_a \to M_b)$ if $a$ not a source

$\psi_\alpha = 0$ for every arrow $\alpha$ of source $a$

<u>EX!!</u> $\operatorname{top} M = 0 \underset{0}{\overset{0}{\leftleftarrows}} K \underset{0}{\overset{0}{\leftleftarrows}} K$