REPRESENTATION STABILITY AND FI-MODULES

PETER WEBB

1. Representation stability

[CF] T. Church, B. Farb, Representation theory and homological stability, Adv Math 245 (2013), 250-314.
introduced 'representation stability'. There is a sequence of groups with natural inclusions $G_n \hookrightarrow G_{n+1}$, a representation V_n of G_n for each i, linear maps $\phi_n : V_n \to V_{n+1}$ so that for all $g \in G_n$ the following diagram commutes:

$$\begin{array}{ccc} V_n & \xrightarrow{\phi_n} & V_{n+1} \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g} \\ V_n & \xrightarrow{\phi_n} & V_{n+1} \end{array}$$

Such a sequence of representations they call *consistent*.

They were interested in groups such as $G_n = SL_n\mathbb{Q}$ and various other classical groups, and also $G_n = S_n$, the symmetric group. The morphism $S_n \to S_{n+1}$ includes S_n as the stabilizer of $n+1$. The sort of representations they consider include $V_n = H^i(Conf_n(M); \mathbb{Q})$ where $Conf_n(M)$ is the *configuration space* of n points on a manifold M. Given a configuration with $n+1$ points we get a configuration with n points by omitting the last point, and this gives a map in cohomology in the opposite direction.

They put three conditions on a consistent sequence to say that it is *representation stable*. We give these conditions in the case of symmetric groups, in characteristic 0. If $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_t)$ is a partition of k, and $(n-k, \lambda_1, \lambda_2, \ldots, \lambda_t)$ is the *padded* partition of n we put $V(\lambda)$ for the simple representation of S_n corresponding to the partition of n. The conditions are that for sufficiently large n:

(1) For sufficiently large n, ϕ_n is injective.
(2) For sufficiently large n, the image of ϕ_n generates V_{n+1} as a representation of S_{n+1}.
(3) For each λ, the composition factor multiplicities $[V_n : V(\lambda)]$ are eventually independent of n.

There is a stronger condition than (3). If (3′) is satisfied we say that the consistent sequence is *uniformly representation stable*:

(3') There exists N, not depending on λ, so that for $n \geq N$ the multiplicities $[V_n : V(\lambda)]$ are independent of n for all λ.

This means that the composition lengths of the V_n are eventually constant. If λ is such that $V(\lambda)$ is not defined at N then $V(\lambda)$ does not appear as a composition factor. It follows that the dimensions of the V_n are eventually polynomial.

2. FI-modules

[CEF] T. Church, J.S. Ellenberg, B. Farb, FI-modules and stability for representations of symmetric groups, preprint Nov 3, 2014
introduced formalism to encode the stability of such sequences. They define FI to be the category of finite sets with morphisms the injections of sets. Recall that a *category* has objects and morphisms, each object has an identity morphism and there is an associative law of composition of morphisms. Up to isomorphism, each object of FI is isomorphic to $[n] = \{1, \ldots, n\}$ for some natural number n.

A *representation* over a commutative ring k of a category $\mathcal{C}$ is a functor $M : FI \to k$-mod. Thus to each object we associate a vector space $M(x)$ and to each morphism a linear map between the vector spaces, so that composition of these linear maps follows the law of composition in the category.

A group may be regarded as a category with just one object where all the morphisms are invertible. In this case a representation of the group regarded as a category is the same thing as a representation of the group in the usual sense.

Given a quiver, we may construct the *free category* on the quiver, and a representation of the free category is the same thing as a representation of the quiver.

If x is an object of a category $\mathcal{C}$ and M is a representation, so that $M(x)$ is a vector space, every endomorphism $\alpha : x \to x$ gives rise to a linear map $M(\alpha) : M(x) \to M(x)$, so that $M(x)$ becomes a representation of the monoid $\mathrm{End}_{\mathcal{C}}(x)$. In FI, the monoid $\mathrm{End}([n])$ is S_n. Thus each representation M of FI includes a representation $M([n])$ of S_n, one for each integer n.

In FI there are $(n+1)!$ morphims $[n] \to [n+1]$ and they lie in a single orbit under composition with S_{n+1}, and $n+1$ orbits under composition on the other side with S_n. Let $i : [n] \to [n+1]$ be the natural inclusion (we could have taken any monomorphism for this with suitable modification). Then for every permutation $\pi \in S_n \leq S_{n+1}$ we have $M(\pi)M(i) = M(i)M(\pi)$ so that $M(i)$ is equivariant for the action of S_n and the sequence $V_n := M([n])$ is consistent.

The following statement appears just before 3.3.2 in [CEF]

Proposition 2.1. *The consistent sequences coming from FI-modules are precisely the ones with the property that if $\sigma \in S_k \leq S_{n+k}$ then σ acts as the identity on the image of V_n in V_{n+k}.*

Proof. One direction follows by functoriality from the identity $\sigma i = i$. □

There is also information in an FI-module that is not encoded in a consistent sequence.

3. Finite generation

Representations of a category $\mathcal{C}$ themselves form an abelian category. Given a representation $M : \mathcal{C} \to k$-mod a subrepresentation is a representation M_1 with $M_1(x) \subseteq M(x)$ for all objects x. There is then a quotient representation M/M_1 with $(M/M_1)(x) := M(x)/M_1(x)$; and so on. There is also an algebra $k\mathcal{C}$ called the *category algebra* with the property that (apart from a finiteness condition when there are infinitely many objects) the representations of $\mathcal{C}$ are the same thing as modules for $k\mathcal{C}$.

A representation M is *finitely generated* if there is a finite subset $S \subseteq \bigsqcup M(x)$ of elements of the values of M so that M is the smallest subfunctor of M containing S.

The following is an immediate deduction from Theorem 1.13 of [CEF].

Theorem 3.1. *Let k be a field of characteristic zero. The following are equivalent for an FI-module V and the corresponding consistent sequence of representations V_n:*

(1) *V is finitely generated;*
(2) *$\{V_n\}$ is uniformly representation stable and each V_n is finite dimensional;*
(3) *$\{V_n\}$ is representation stable and each V_n is finite dimensional.*

Proof. (1) implies (2) is substantial and (2) implies (3) is immediate. We do (3) implies (1) only. Suppose that $\{V_n\}$ is representation stable, so that there exists N for which the image of ϕ_n generates V_{n+1} as a kS_{n+1}-module when $n \geq N$. This means that V is generated by its values at objects $[n]$ with $n \leq N$. Since those spaces are all finite dimensional, they are finitely generated and (1) follows. □

A characteristic zero version of the following theorem appears as Theorem 1.3 of [CEF]. Over Noetherian rings it was proved in [CEFN]. The essential categorical properties used were extracted in [GL] and a proof given in that generality. Such an approach was also taken in [SS]. Two further proofs were given by Liping Li in [L].

Theorem 3.2. *Let k be a Noetherian ring. Then FI-modules are Noetherian.*

Proof. It comes down to examining the linearized representable functors $P_{[n]}$. For any object x in a category $\mathcal{C}$ we have a linearized representable functor $P_x : \mathcal{C} \to k$-mod given by $P_x(y) = k\,\mathrm{Hom}_{\mathcal{C}}(x, y)$. By Yoneda's lemma these are projective. P_x is generated by its value at x. Every finitely generated representation of $\mathcal{C}$ is an image a finite direct sum of these. It suffices to show that the P_x are Noetherian. □

4. Bibliography

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E-mail address: webb@math.umn.edu

School of Mathematics, University of Minnesota, Minneapolis, MN 55455, USA