Quiver algebras

Throughout: $K$ is an algebraically closed field.
Modules will be left-modules.

Let $Q$ be a quiver (directed graph, loops & multiple edges are allowed).

**Definition**

The quiver algebra $KQ$ is the $K$-vector space with basis the set of paths of length $\geq 0$ in $Q$.

$$
\beta \cdot \alpha := \begin{cases} 
\sum \beta \alpha_i & (\alpha \text{ followed by } \beta) \\
0 & \text{if these paths can concatenate} \\
0 & \text{otherwise}
\end{cases}
$$

**Examples**

**Remark**

This is the category algebra of the free category generated by $Q$.

**Example**

$$Q = 1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3 \xrightarrow{\gamma} 4$$

$$KQ = \text{span} \{ \alpha, \beta, \gamma, \beta \alpha, \gamma \beta, \gamma \beta \alpha, \}_{3} \{ \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4 \}
$$

Paths of length 0:

$$\vec{\beta} + 4 \gamma \alpha \vec{x} = \gamma \beta \alpha + 4 \gamma \beta \alpha = \gamma \beta \alpha.$$
Examples

b) \( Q = 1 \otimes \mathbb{C} \)

\[ kQ = \text{span} \{ 1, \alpha, \alpha^2, \ldots \} \]

\[ = k[\alpha] \]

c) \( Q = \begin{array}{c}
1 \\
\alpha \\
\alpha^2 \\
\vdots
\end{array} \)

\[ kQ = k\langle t_1, \ldots, t_n \rangle \]

\[ \text{noncommuting variables} \]

Remark: \( kQ \) is not always finite-dimensional, but there is a sufficient condition on two-sided ideals \( I \) such that \( kQ/I \) is finite-dimensional.

Let \( R^*_Q \) be the two-sided ideal spanned by all paths of length \( \geq \frac{1}{\epsilon} \). Then \( (R^*_Q \text{ has all paths of length } \geq \epsilon c) \).

(Prop) \( R^n_Q \leq I \leq R^2_Q \Rightarrow \frac{kQ}{I} \) is finite-dimensional for some \( n \).
In this case we say \( I \) is admissible. (It is convenient to restrict our attention to admissible ideals.)

\[ (\text{finite generated}) \]

\textbf{Question:} What do modules over \( KQ/I \) look like?

They are obtained as follows:

\begin{enumerate}
  \item Draw \( Q \).
  \item Place a f.d. \( K \)-vector space at each vertex of \( Q \).
  \item Define linear maps at each arrow which satisfy the relations in \( I \).
\end{enumerate}

\[ Q = \begin{array}{c}
  0 \\
  \circ \\
  2 \\
  \circ \\
  \alpha, \gamma
\end{array} \quad K = \begin{array}{c}
  0 \\
  \circ \\
  2 \\
  \circ \\
  \alpha, \gamma
\end{array} \quad A := KQ / \langle \alpha \beta \gamma \rangle
\]

Let \( M = K \leftarrow K^2 \leftarrow K \leftarrow K \). (Indeed, \( fgf^2 f^4 g^2 = 0 \)).

This is a 5-dimensional space, where for example \( \sigma \) acts as the 5x5 matrix

\[
\begin{bmatrix}
  0 & 0 & 1 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]
The simple $KQ/I$-modules are

\[ 1'' = L(1) \cdot S(1) = K \to 0 \to 0 \to 0 \]

\[ S(2) = 0 \to K \to 0 \to 0 \]

\[ S(3) = 0 \to 0 \to K \to 0 \]

\[ S(4) = 0 \to 0 \to 0 \to 0 \to K \]

So $KQ/I$ is basic ($\dim S = 1 \forall S$).

The semisimple $KQ/I$-modules are of the form

\[ K^{n_1} \leftarrow K^{n_2} \leftarrow K^{n_3} \leftarrow K^{n_4} \]

(Place any vector space at each vertex, define all maps to be zero.)

The regular representation ($KQ/I$ as a left $KQ/I$-module, given by left multiplication).

Is given as follows.

1. At each vertex, place a copy of $K$ for each path ending there
2. Define maps according to left multiplication.
Ex: \( A = \mathbb{K} \mathfrak{H} \) where \( Q = \begin{pmatrix} 1 & x & \beta & \gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \ I = \{ \mathfrak{pa} \} \).

\[
A A = \begin{bmatrix}
K^3 & [0]
\end{bmatrix}
\begin{bmatrix}
K^2 & [0]
\end{bmatrix}
\begin{bmatrix}
K & [0]
\end{bmatrix}
\]

\[
\begin{bmatrix}
\text{span}(e_1, \alpha) \\
\text{span}(e_2, \beta, \gamma)
\end{bmatrix}
\begin{bmatrix}
\text{span}(e_3, \rho, \pi)
\end{bmatrix}
\begin{bmatrix}
\text{span}(e_4, \mu)
\end{bmatrix}
\begin{bmatrix}
\text{span}(e_5)
\end{bmatrix}
\]

The direct summands are the projective indecomposables:

\[
P(4) = \begin{bmatrix} 0 \\ K \\ K \\ K \end{bmatrix}
\]

\[
P(3) = \begin{bmatrix} K \\ K \\ K \\ 0 \end{bmatrix}
\]

\[
P(2) = \begin{bmatrix} K \\ K \\ 0 \\ 0 \end{bmatrix}
\]

\[
P(1) = \begin{bmatrix} K \\ 0 \\ 0 \\ 0 \end{bmatrix}
\]
The injective indecomposables are the direct summands of $\text{Hom}(A, K)$.

$\text{Hom}(A, K)$ is constructed as follows:

1. At vertex $i$, place a copy of $K$ for each path beginning there.
2. Define maps according to "the dual of right multiplication."

In our example, $\text{Hom}(A, K) = 0 \to 0 \to 0 \to _{\text{K}_1, \text{K}_2, \text{K}^\beta} \to 0$):

- $\text{I}(4)$
- $\text{I}(3)$
- $\text{I}(3)$
- $\text{I}(2)$
- $\text{I}(1)$

Here, e.g. $\text{I}(1) = \text{span} \{ [\text{E}], [\text{Y}], [\text{K}^\beta] \}$, but e.g. $

\beta \cdot [\text{K}^\beta] = [\text{Y}]$

(un-multiply by $\beta$ since the path begins with $\beta$).

Yep, it's a left action.
The radical?

\[
\text{rad}(kQ/I) = R_k Q
\]

\[
\text{rad}^k(kQ/I) = R_k Q^k
\]

And, as before, \( \text{rad}^k(M) = \text{rad}^k(kQ/I) M = R_k Q^k M \).

"vectors acted on by paths of length \( k \)"

So \( k \)?

\[ \text{soc}^k(M) = \text{"vectors that are annihilated by every path of length } k \text{"} \]

\[
\text{Note! }\operatorname{dim} \left( \frac{R_k Q}{R_k Q^2} \right) = \text{total } \# \text{ of edges in } Q.
\]

\[
\operatorname{dim} E_i \left( \frac{R_k Q}{R_k Q^2} \right) E_j = \text{total } \# \text{ of edges from } i \text{ to } j.
\]
Main structure theorem: Let $A$ be a f.d. K-algebra which is basic (dim $S = 1$ if $S$ simple) and connected ($A \cong A_1 \times A_2$ only trivially).

Then there is a quiver $Q$ such that $A \cong KQ$ for some admissible ideal $I$.

Proof sketch. Let $e_1, \ldots, e_n$ be a complete system of pairwise orthogonal idempotents.

Define $Q$ as follows:

- Vertices $1, \ldots, n$ in bijection with idempotents.
- The number of edges from $i$ to $j$ is $\dim_k \left( e_i \frac{\text{rad} A}{\text{rad}^2 A} e_j \right)$

We get a quiver algebra $KQ$ and a map $KQ \xrightarrow{\phi} A$ sending $e_i \mapsto e_i$

(each edge $x$) $1 \mapsto (\text{the corresponding basis vector of } \frac{\text{rad} A}{\text{rad}^2 A})$.

This is well-defined and surjective!

And her $\phi$ is an admissible 2-sided ideal.