Homework #5 for MATH 8301: Manifolds and Topology

October 2, 2017

Due Date: Monday 9 October in class.

1. Let G be a group, and define $\mathcal{C}$ to be the category with a single object $*$ and

   $$\mathrm{Hom}_{\mathcal{C}}(*, *) = G.$$

   The identity element of G gives the identity morphism $i_* : * \to *$, and composition in this category comes from multiplication in G.

   Let k be a field, and let Vect_k be the category of vector spaces over k: objects are vector spaces, and morphisms are linear maps.

   (a) Recall that a *representation* of a group G is an action of G on a vector space V where each element $g \in G$ acts on V through linear maps. Let $F : \mathcal{C} \to \mathrm{Vect}_k$ be a functor. Show that $V := F(*)$ is a representation of G.

   (b) Conversely, for any representation V of G over a field k, construct a functor $F : \mathcal{C} \to \mathrm{Vect}_k$ with $F(*) = V$.

2. Let $\mathcal{C}$ be a category. A morphism $f : X \to Y$ is said to be an *isomorphism* if there exists a morphism $g : Y \to X$ with the property that $f \circ g = \mathrm{id}_Y$ and $g \circ f = \mathrm{id}_X$.

   (a) Let Set be the category of sets. Show that a map $f : X \to Y$ is an isomorphism in Set if and only if it is a bijection.

   (b) Let Top be the category of topological spaces; show that $f : X \to Y$ is an isomorphism in Set if and only if it is a homeomorphism.

   (c) Let hTop be the homotopy category of topological spaces: objects in hTop are topological spaces, and

   $$\mathrm{Hom}_{\mathrm{hTop}}(X, Y) = \{f : X \to Y \text{ continuous}\}/ \simeq$$

   where $f \simeq g$ if f is homotopic to g. Show that $f : X \to Y$ is an isomorphism in hTop if and only if it is a homotopy equivalence.

   (d) Show that every morphism in the the category $\mathcal{C}$ of problem 1 is an isomorphism.

(e) Let $F : \mathcal{C} \to \mathcal{D}$ be a functor, and let $f : X \to Y$ be an isomorphism in $\mathcal{C}$. Show that $F(f)$ is an isomorphism in $\mathcal{D}$.

3. For a topological space X, let $\pi_0(X)$ denote the set of path components of X; this is the quotient set of X under the relation $x \sim y$ if there exists a path in X from x to y.

(a) For a continuous map $f : X \to Y$, let $\pi_0(f) : \pi_0(X) \to \pi_0(Y)$ denote the function $\pi_0(f)([x]) = [f(x)]$, where $[-]$ denotes equivalence classes under $\sim$. Show that $\pi_0(f)$ is well-defined.

(b) Show that the operations π_0 (defined above on spaces and continuous maps) defines a functor $\pi_0 : \mathrm{Top} \to \mathrm{Set}$.

(c) Show that in fact π_0 also defines a functor $\pi_0 : \mathrm{hTop} \to \mathrm{Set}$ by the same formula.

(d) Prove that if X is homotopy equivalent to Y, then $\#\pi_0(X) = \#\pi_0(Y)$. That is, the cardinality of the set of path components of X and Y agree. **Hint:** Problem 2(e) may be helpful.