Homework #8 for MATH 8301: Manifolds and Topology

October 31, 2017

Due Date: Monday 6 November in class.

1. Let $z_1, \ldots, z_n \in \mathbb{C}$ be n distinct points (i.e., $z_i \neq z_j$ for $i \neq j$), and let $f(z) = (z - z_1) \cdots (z - z_n)$. Define

$$Y := \{(z, w) \mid w^2 = f(z)\}.$$

The space Y is a Riemann surface, known as a *hyperelliptic curve.* Define a function $p : Y \to \mathbb{C}$ by $p(z, w) = z$; this is an example of a *branched covering* of $\mathbb{C}$, and $\{z_1, \ldots, z_n\}$ is the *branch locus.* Define $X = \mathbb{C} \setminus \{z_1, \ldots, z_n\}$, and

$$\overline{X} := Y \setminus p^{-1}(\{z_1, \ldots, z_n\}) = Y \setminus \{(z_1, 0), \ldots, (z_n, 0)\}.$$

Let $p : \overline{X} \to X$ the restriction of p to these subspaces.

(a) Notice that X is the subspace of $\mathbb{C}$ on which $f(z) \neq 0$. Define four subspaces of X by

$$R_+ = \{z \in X \,|\, \Re(f(z)) > 0\}, \quad R_- = \{z \in X \,|\, \Re(f(z)) < 0\},$$
$$I_+ = \{z \in X \,|\, \Im(f(z)) > 0\}, \quad I_- = \{z \in X \,|\, \Im(f(z)) < 0\}.$$

Here, if $z = a + bi$, $\Re(z) = a$ denotes the real part of z, and $\Im(z) = b$ is the imaginary part. Using these subspaces, show that $p : \overline{X} \to X$ is a covering space.

(b) Let $f : \overline{X} \to \overline{X}$ be the function $f(z, w) = (z, -w)$. Show that f is a homeomorphism, and that $p \circ f = p$. Such maps are called *deck transformations* or *automorphisms* the covering space.

(c) Besides the identity of $\overline{X}$, are there any other automorphisms of $p : \overline{X} \to X$?

(d) For simplicity, let's take $n = 2$, and define $z_1 = 0$, and $z_2 = 2$. Let $x_0 = 1 \in X \subseteq \mathbb{C}$ be a basepoint for X. What is $p^{-1}(x_0)$?

(e) The fact that $p \circ f = p$ implies that f permutes elements of $p^{-1}(x_0)$. Compute this permutation.

(f) Define a loop $\gamma : [0,1] \to X$ based at x_0 by $\gamma(t) = e^{2\pi it}$. By the path lifting property of covering spaces, for each element $x \in p^{-1}(x_0)$, there is a unique lift of γ to a path $\overline{\gamma}_x : [0,1] \to \overline{X}$ (with $p \circ \overline{\gamma}_x = \gamma$) based at x (i.e., $\overline{\gamma}_x(0) = x$) . Find a formula for all of these lifted paths (hint: there are only two lifts).

(g) In the previous part, the fact that $p \circ \overline{\gamma}_x = \gamma$ implies that $\overline{\gamma}_x(1) \in p^{-1}(x_0)$. We may define a new permutation of $p^{-1}(x_0)$ by the formula

$$x \mapsto \overline{\gamma}_x(1).$$

Compute this permutation. How does it relate to the one coming from part (e)?

2. Let $L_1, \ldots, L_n$ be n distinct lines in $\mathbb{R}^3$ passing through the origin. Compute the fundamental group of

$$\mathbb{R}^3 \setminus \bigcup_{i=1}^{n} L_i.$$

3. The *join* $X * Y$ of two (nonempty) spaces X and Y is the quotient of the product $X \times Y \times I$ by the equivalence relation

$$(x, y, 0) \sim (x', y, 0) \text{ and } (x, y, 1) \sim (x, y', 1)$$

for all $x, x' \in X$ and $y, y' \in Y$. If X is path connected, show that $X * Y$ is simply connected.