Homework #4 for MATH 8302: Manifolds and Topology II

April 6, 2018

Due Date: Friday 13 April in class.

1. Using the results of the generalized Stokes theorem, prove the following classical theorems in two and three dimensions:

   (a) (Green's theorem) Let W be a compact domain in $\mathbb{R}^2$ with smooth boundary ∂W which we will take to be a simple closed curve $\gamma : S^1 \to \mathbb{R}^2$. Let f and g be smooth functions on $\mathbb{R}^2$; then

   $$\int_{\partial W} f\,dx + g\,dy = \int_W \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dx\,dy.$$

   (b) (Divergence Theorem) Let W be a compact domain in $\mathbb{R}^3$ with smooth boundary ∂W, a smooth surface in $\mathbb{R}^2$. Let $F = (f_1, f_2, f_3)$ be a smooth vector field on W. Show that

   $$\int_W (\nabla \cdot F)\,dx\,dy\,dz = \int_{\partial W} (u \cdot F)\,dA$$

   where u is the outward-pointing *unit* normal vector on ∂W, and dA is the area form on dW (see pages 170-171 of G&P as well as exercises 13, 14 in section 4.4 for details on this).

   (c) (Stokes' theorem) Let S be a compact surface in $\mathbb{R}^3$ with boundary, $F = (f_1, f_2, f_3)$ be a smooth vector field on S. Prove

   $$\int_S (\nabla \times F) \cdot u\,dA = \int_{\partial S} f_1\,dx_1 + f_2\,dx_2 + f_3\,dx_3.$$

2. This problem is about how integration interacts with bordism and cohomology.

   (a) Let $f : X \to Y$ be a smooth map of k-manifolds, $\omega \in \Omega^k(Y)$. Assume that there is some $(k+1)$-manifold W with $\partial W = X$, and that f extends over W to give $F : W \to Y$ (we say (W, F) is a *null-bordism* of (X, f)). Show that $\int_X f^*\omega = 0$.

(b) Let X_1 and X_2 be cobordant k-manifolds: there is a $(k+1)$-manifold W with $\partial W = X_1 \sqcup X_2$. Let $\omega \in \Omega^k W$; show

$$\int_{X_1} \omega = \int_{X_2} \omega.$$

3. If X is a k-manifold, $Z \subseteq X$ a p-dimensional submanifold, and $\omega, \omega' \in \Omega^p(X)$ *cohomologous* p-forms, show that

$$\int_Z \omega = \int_Z \omega'.$$

Conclude that the operation $\int_Z : H^p(X) \to \mathbb{R}$ is well-defined.