Week 1: Configuration spaces and their many guises

September 14, 2015

The subject of this course will be, loosely, to survey what algebraic topology can say about configuration, moduli, and function spaces, most particularly with an eye towards their use in algebraic geometry and geometric topology.

Definition 1. Let X be a topological space, and $n \in \mathbb{Z}_{\geq 0}$. Define the n^{th} ***ordered configuration space*** of X as

$$\mathrm{PConf}_n(X) := \{(x_1, \dots, x_n) \in X^{\times n} \mid \text{if } x_i = x_j \text{, then } i = j\}$$

This is topologized as a subspace of $X^{\times n}$. Many authors will notate this space as $F(X, n)$.

We note that the symmetric group S_n acts on $X^{\times n}$ by permuting coordinates. The restriction of this action to $\mathrm{PConf}_n(X)$ is free, as we have removed the fixed points.

Definition 2. The n^{th} ***unordered configuration space*** of X is

$$\mathrm{Conf}_n(X) := \mathrm{PConf}_n(X)/S_n.$$

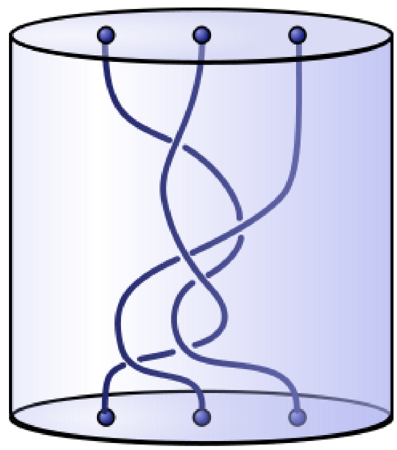
This is sometimes notated $B(X, n)$ in the literature. The quotient map $\mathrm{PConf}_n(X) \to \mathrm{Conf}_n(X)$ is a Galois covering, with deck transformation group S_n.

We will explore several manifestations of configuration spaces in mathematics, and will mainly focus on the case that $X = \mathbb{C}$ is the complex plane.

The fundamental group $\pi_1 \mathrm{Conf}_n(\mathbb{C})$ is the n^{th} braid group B_n

Recall that the ***braid group*** B_n is defined to be the set of isotopy classes of braids in $\mathbb{R}^3$ connecting two planes, with multiplication given by stacking braids on top of each other. The ***pure braid group*** PB_n is the subgroup of B_n where each strand begins and ends at the same point.

It's not hard to convince yourself that $\pi_1 \mathrm{Conf}_n(\mathbb{C}) = B_n$, and $\pi_1 \mathrm{PConf}_n(\mathbb{C}) = PB_n$. A loop in $\mathrm{Conf}_n(\mathbb{C})$ is a path beginning and ending at the same configuration; this may be regarded as an n-tuple of paths beginning and ending at the same n-tuple of distinct points in $\mathbb{C}$. The graph of such a family of paths is precisely an n-strand braid. A homotopy of loops gives rise to a homotopy of such paths. It must in fact be an isotopy, since at all times the points in the strands are required to be distinct.



Some linguistics: B_n is the ***Artin group*** associated to the ***Coxeter group*** S_n. It has the presentation

$$B_n = \langle \sigma_1, \ldots, \sigma_{n-1} \mid \sigma_i\sigma_j = \sigma_j\sigma_i \text{ if } |i-j| > 1, \text{ and } \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \rangle.$$

Here σ_i is the braid which swaps the i^{th} and $i+1^{\text{st}}$ points in the configuration via a simple crossing (say with the right-going strand in front). The symmetric group S_n is obtained from this by adding the relations that all generators are involutions.

Squarefree polynomials

Let X_n be the algebraic variety of squarefree polynomials; over a field k,

$$X_n(k) = \{\text{polynomials } p(z) \in k[z] \text{ which are monic of degree } n\}.$$

The set of all monic polynomials of degree n is affine space of dimension n; a natural isomorphism is given by the association

$$\left[p(z) = z^n + a_1 z^{n-1} + \cdots + a_{n-1}z + a_n\right] \leftrightarrow (a_1, \ldots, a_n) \in \mathbb{A}^n.$$

Consequently X_n may be identified as an open subvariety of $\mathbb{A}^n$. In fact, it is the complement of the discriminant locus: if $\{z_1, \ldots, z_n\}$ are the roots of p over an algebraically closed field, the discriminant of p is

$$\Delta(p) := (-1)^{\binom{n}{2}} \prod_{i \neq j} (z_i - z_j).$$

Evidently $\Delta(p)$ vanishes precisely when p has repeated roots or equivalently when p is not squarefree (over a field containing all the roots). Thus we may define X_n as the open variety $\mathbb{A}^n \setminus \Delta^{-1}(0)$.

Over the complex numbers, we then get obtain an identification

$$X_n(\mathbb{C}) = \mathrm{Conf}_n(\mathbb{C})$$

via the bijection which carries $p(z)$ to its set of roots. Since p is squarefree, there are precisely n of these; furthermore, they come without a natural ordering (giving an element of $\mathrm{Conf}_n(\mathbb{C})$, rather than $\mathrm{PConf}_n(\mathbb{C})$).

Genus 0 moduli spaces

We define the ***moduli space of*** n ***points in genus*** 0 as

$$\mathcal{M}_{0,n} := \{(X, \underline{z}) \mid \underline{z} \subseteq X \text{ is an } n\text{-tuple of distinct points in } X\text{, a projective curve of genus } 0\}/\cong.$$

However, every projective curve of genus 0 is isomorphic to $\mathbb{P}^1$, so we may rewrite this as

$$\mathcal{M}_{0,n} = \mathrm{PConf}_n(\mathbb{P}^1)/\operatorname{Aut}(\mathbb{P}^1).$$

We recall that $\operatorname{Aut}(\mathbb{P}^1)$ is the group of Möbius (or linear fractional) transformations. Furthermore, for any three distinct points $z_1, z_2, z_3 \in \mathbb{P}^1$, there is a unique element $\mu \in \operatorname{Aut}(\mathbb{P}^1)$ with $\mu(z_1) = 0$, $\mu(z_2) = 1$, and $\mu(z_3) = \infty$. Thus for $n \geq 3$, there is an isomorphism

$$\mathcal{M}_{0,n} \cong \mathrm{PConf}_{n-3}(\mathbb{A}^1 \setminus \{0,1\})$$

which carries $(z_1, \ldots, z_n)$ to $(\mu(z_4), \ldots, \mu(z_n))$. Alternatively, we may write this as

$$\mathcal{M}_{0,n} \cong \mathrm{PConf}_{n-1}(\mathbb{A}^1)/\operatorname{Aff}(\mathbb{A}^1)$$

where the ***affine group*** $\operatorname{Aff}(\mathbb{A}^1) < \operatorname{Aut}(\mathbb{P}^1)$ is the subgroup fixing ∞.

The higher genus analogues $\mathcal{M}_{g,n}$ are defined similarly. However, since there is a finite dimensional moduli space of isomorphism classes of curves of genus g (rather than the unique one in genus 0), this space cannot simply be defined as a configuration space. Over $\mathbb{C}$, this extra data may be encoded in the Teichmüller space of hyperbolic metrics on the curve.

Homology representations of S_n

Since S_n acts on $\mathrm{PConf}_n(\mathbb{C})$, it naturally is equipped with a linear action on $H_*(\mathrm{PConf}_n(\mathbb{C}))$. It is a natural question to describe the structure of this S_n representation. This is possible (in a certain sense) using homotopy theory.

Definition 3. A *Gerstenhaber algebra* A_* is a graded abelian group equipped with two binary operations

$$- \cdot - : A_m \otimes A_n \to A_{m+n} \quad \text{and} \quad [-,-] : A_m \otimes A_n \to A_{m+n+1}$$

making A simultaneously into a graded commutative, associative ring and a graded Lie algebra (with bracket of degree 1); these structures interact via a Leibniz rule (i.e., the bracket is a derivation of the product in each variable). The Lie algebra grading is shifted by 1, which makes a mess of the signs in the axioms:

- $[a, b] = (-1)^{(|a|+1)(|b|+1)}[b, a]$,
- $[a, [b, c]] = [[a, b], c] + (-1)^{(|a|+1)(|b|+1)}[b, [a, c]]$,
- $[a, b \cdot c] = [a, b] \cdot c + (-1)^{(|a|+1)|b|}b[a, c]$.

Define $\mathrm{Gerst}_i(n)$ to be the space of $\mathbb{Z}$-linear combinations of n-ary operations of degree i on Gerstenhaber algebras formed from combinations of the product and bracket. For instance, $\mathrm{Gerst}_0(1) = \mathbb{Z}\{\text{id}\}$ is generated by the identity (a 1-fold product), and there are no other unary operations. Using dummy variables, the binary operations are

$$\mathrm{Gerst}_0(2) = \mathbb{Z}\{a \cdot b\} \quad \text{and} \quad \mathrm{Gerst}_1(2) = \mathbb{Z}\{[a, b]\}$$

and the ternary operations are

$$\begin{aligned} \mathrm{Gerst}_0(3) &= \mathbb{Z}\{a \cdot b \cdot c\}, \\ \mathrm{Gerst}_1(3) &= \mathbb{Z}\{a \cdot [b, c], b \cdot [c, a], c \cdot [a, b]\} \\ \mathrm{Gerst}_2(3) &= \mathbb{Z}\{[a, [b, c]], [[a, b], c]\}. \end{aligned}$$

The axioms defining Gerstenhaber algebras ensures that all other ternary operations are linear combinations of these. Notice that S_n acts on $\mathrm{Gerst}_i(n)$ by permutation of the inputs $a_1, \ldots, a_n$.

Theorem 4 (F. Cohen). *There is an isomorphism* $\mathrm{Gerst}_i(n) \cong H_i(\mathrm{PConf}_n(\mathbb{C}))$ *of* S_n*-representations.*

This is just a part of the larger story of the little disks operads and iterated loop spaces. Which brings us to:

Function spaces

If X and Y are topological spaces, we will write $\mathrm{Map}(X, Y)$ for the topological space of all continuous functions from X to Y, equipped with the k-ification[1] of the compact-open

[1] Recall that for a topological space Z, the k-ification kZ has the same underlying set, equipped with the compactly generated topology (from the original topology on Z). That is: $W \subseteq Z$ is closed in kZ if and only if $g^{-1}(W)$ is closed for any map $g : K \to Z$ where K is compact, and g is continuous with respect to the original topology on Z.

topology. If X and Y have basepoints, we will write $\mathrm{Map}_*(X, Y)$ for the subspace of maps which carry the basepoint of X to that of Y.

If $X = S^k$, we write $\Omega^k Y = \mathrm{Map}_*(S^k, Y)$. Recall that $\pi_0 Z$ denotes the set of path components in Z. A path in a function space from f to g is the same data as that of a homotopy between f and g. Thus $\pi_0 \mathrm{Map}(X, Y)$ is the set of homotopy classes of maps from X to Y. In particular, $\pi_0 \Omega^k Y = \pi_k Y$ is the k^{th} homotopy group of Y. It goes without saying that these groups are the central object of study in homotopy theory; thus we are very interested in the spaces $\Omega^k Y$. For a class $\alpha \in \pi_k Y$, we will write $\Omega^k_\alpha Y \subseteq \Omega^k Y$ for the component indicated by α.

Perhaps surprisingly, the homology of $\mathrm{Map}(X, Y)$ can sometimes described using configuration spaces. For instance, there is an "electrostatic map" (introduced in this form by Segal)

$$e : \mathrm{Conf}_n(\mathbb{R}^k) \to \Omega^k_n S^k$$

(here Ω^k_n indicates that the degree of these maps is n). One may assign an electric charge to each element of a configuration $\underline{x} := (x_1, \ldots, x_n)$ in $\mathbb{R}^k$. The associated electric field (with poles at the x_i) may be regarded as a function $e(\underline{x})$ from $\mathbb{R}^k$ to $S^k = \mathbb{R}^k \cup \{\infty\}$ which extends naturally over infinity.

The following is a sample of many such results due to May, Boardman-Vogt, Segal, and McDuff, amongst others:

Theorem 5. *The induced map* $e_* : H_p \mathrm{Conf}_n(\mathbb{R}^k) \to H_p \Omega^k_n S^k$ *is always injective, and an isomorphism in dimensions* $p \leq n/2$.

Originally, this result (and many others like it) were used to flow information from left to right: one computes the homology of configuration spaces using devious tricks (which we will explore in the coming weeks), and concludes something about the homology of function spaces (which was the original goal, for a better understanding of homotopy groups). The flow of information can, of course, go in the other direction: we may be able to study the homology of function spaces (for instance, using rational homotopy theory) in order to conclude something about the homology of configuration spaces. The previous sections suggest that mathematicians in other fields may have reason to be interested in these computations.

Gerstenhaber structure on homology of function spaces

We recall that the homotopy groups $\pi_k Y$ are *groups* – this is perhaps most familiar when $k = 1$: the multiplication on the fundamental group is given by concatenation of loops. One might wonder if this algebraic structure is reflected in the topology of the space $\Omega^k Y$ or its homology in some fashion. This is in fact the case; let us focus on the case $k = 2$ for pictorial simplicity:

Theorem 6 (F. Cohen). *For any space X, $H_*\Omega^2 X$ is a Gerstenhaber algebra.*

We will give a proper proof of this result (and its generalization to $\Omega^k X$) later, but sketch a picture proof here. We will construct a family of maps of spaces

$$\mathrm{PConf}_n(\mathbb{C}) \times (\Omega^2 X)^{\times n} \to \Omega^2 X \tag{1}$$

for each n. Applying homology and invoking Theorem 4, we get maps

$$\gamma_n : \mathrm{Gerst}_*(n) \otimes (H_*\Omega^2 X)^{\otimes n} \to H_*\Omega^2 X.$$

By their very nature[2], these maps define the structure of a Gerstenhaber algebra on $H_*\Omega^2 X$, since $\mathrm{Gerst}_*(n)$ parameterizes the set of n-ary operations intrinsic to all Gerstenhaber algebras, and γ_n gives a specific presentation of these n-ary operations.

How would we obtain maps of the form (1)? First, rewrite the double loop space as

$$\Omega^2 X = \mathrm{Map}((D^2, \partial D^2), (X, *))$$

Furthermore, we will confuse $\mathbb{C}$ with the interior of D^2, to which it is homeomorphic. Given an ordered configuration $\underline{z} = (z_1, \ldots, z_n)$ and a family of maps $f_i : (D^2, \partial D^2) \to (X, *)$, the map $g = \gamma_n(\underline{z}, f_1, \ldots f_n)$ is gotten in the following fashion:

1. Draw little non-overlapping circles around each z_i, say with radius equal to $\frac{1}{3}$ of the minimum distance between the z_i.
2. Define $g : D^2 \to X$ to be constant at the basepoint $*$ on the complement of these little disks.
3. Note that there is a "standard" homeomorphism h_i from D^2 to each little disk centered on z_i given by translation and scaling.
4. On the interior of the i^{th} little disk (centered on z_i), let g be given by $f_i \circ h_i^{-1}$. Note that since f_i is constant on the boundary of D^2, the resulting function g is continuous.

There is a lot of machinery developed to utilize these sorts of constructions; this will lead us into *operads.*

[2]We are, of course, neglecting the compatibility of these maps with the axioms of a Gerstenhaber algebra; until a proper proof is given, we encourage the reader to employ the "optimism principle" (thanks to Dylan Wilson for the terminology): this says that in algebraic topology, if one can write down a map, then it's the right map.

Squarefree polynomials, again

Return for a moment to the variety X_n of monic, squarefree, degree n polynomials. We notice that over a finite field $\mathbb{F}_q$, $X_n(\mathbb{F}_q)$ is a finite set. It is classical to enumerate this set:

$$\#X_n(\mathbb{F}_q) = q^n - q^{n-1},\ n > 1.$$

A proof is as follows: write $\zeta(t)$ for the generating function in which the coefficient of t^n is the total number of polynomials of degree n:

$$\zeta(t) = \sum_{n=0}^{\infty} q^n t^n = \frac{1}{1-qt}$$

Let $f(t)$ be the generating function for the squarefree polynomials: $f(t) = \sum \#X_n(\mathbb{F}_q)t^n$. Then $\zeta(t) = f(t)\zeta(t^2)$, reflecting the decomposition of an arbitrary monic polynomial into the product of a squarefree polynomial and a square. Solving for $f(t)$ we have

$$f(t) = \frac{\zeta(t)}{\zeta(t^2)} = \frac{1-qt^2}{1-qt} = (1-qt^2)\sum_{n=0}^{\infty} q^n t^n = 1 + qt + \sum_{n=2}^{\infty}(q^n - q^{n-1})t^n,$$

giving the result.

We may obtain this result through cohomological methods, and the Grothendieck-Lefschetz fixed point theorem. While this is of course a ludicrously heavy-handed way to reprove the above argument, such techniques may avail us in similar questions without as classical a solution.

One consequence of Theorem 5 is that, for $n > 1$,

$$H_p(\mathrm{Conf}_n(\mathbb{C}), \mathbb{Q}) \cong H_p(\Omega^2_n S^2, \mathbb{Q}) = \begin{cases} \mathbb{Q}, & p = 0, 1; \\ 0, & \text{else.} \end{cases}$$

One can prove this using rational homotopy theory, for instance. Alternatively, we may employ Theorem 4, and compute the S_n-invariants of the Gerstenhaber operad:

$$H_*(\mathrm{Conf}_n(\mathbb{C}), \mathbb{Q}) \cong H_*(\mathrm{PConf}_n(\mathbb{C}), \mathbb{Q})^{S_n} = \mathrm{Gerst}_*(n)^{S_n}$$

Poincaré duality implies that the compactly supported cohomology $H_c^{2n-p}(\mathrm{Conf}_n(\mathbb{C}), \mathbb{Q})$ is concentrated at $p = 0, 1$, where it is $\mathbb{Q}$. Using Artin's comparison theorem, we may conclude a similar fact for the étale cohomology; for ℓ coprime to q:

$$H^{2n-p}_{c,\acute{e}t}(X_n|_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell) = \begin{cases} \mathbb{Q}_\ell, & p = 0, 1; \\ 0, & \text{else.} \end{cases}$$

One may establish that the Frobenius operator Frob_q on this vector space acts with trace q^n in the case $p = 0$ and q^{n-1} when $p = 1$. Applying the Grothendieck-Lefschetz fixed point theorem, we alternatively conclude that $\#X_n(\mathbb{F}_q) = q^n - q^{n-1}$.