Symmetric products

Definition 1. For a space $X$ and nonnegative integer $n$, we define $\text{Sym}^n(X)$ or $\text{SP}^n(X)$ to be the $n$-fold symmetric product of $X$:

$$\text{Sym}^n(X) = X^n / S_n,$$

where $S_n$ acts via permutation of coordinates.

The configuration space $\text{Conf}_n(X) \subseteq \text{Sym}^n(X)$ embeds as the open subset of $n$-tuples of distinct elements. We note that, in contrast to configuration spaces, symmetric products are functorial for all continuous maps (not just injective ones). Consequently, the homotopy type of $\text{Sym}^n(X)$ depends only upon the homotopy type of $X$. In particular, if $X$ is contractible, $\text{Sym}^n(X) \simeq *$ is, too.

In what is to come, it will be helpful to have the stronger identification of these spaces up to homeomorphism in certain cases.

Proposition 2. The following hold for $\mathbb{R}$:

1. The configuration space $\text{Conf}_n(\mathbb{R})$ is homeomorphic to the interior of the closed $n$-dimensional simplex, $\Delta^n$. Note that this in turn is homeomorphic to $\mathbb{R}^n$.

2. Similarly, $\text{Sym}^n(\mathbb{R})$ is homeomorphic to the complement of a two faces in $\Delta^n$.

Proof. Using the homeomorphism $\mathbb{R} \cong (0, 1)$, we of course have $\text{Conf}_n(\mathbb{R}) \cong \text{Conf}_n((0, 1))$. Any $n$-tuple of distinct points $(y_1, \ldots, y_n)$ in $(0, 1)$ has a unique reordering $(y_{\sigma(1)}, \ldots, y_{\sigma(n)})$ with the property that $y_{\sigma(i)} < y_{\sigma(i+1)}$. So:

$$\text{Conf}_n(\mathbb{R}) \cong \{(x_1, \ldots, x_n) \mid 0 < x_1 < x_2 < \cdots < x_n < 1\}.$$
We recall that $\Delta^n = \{(z_0, \ldots, z_n) \in \mathbb{R}^n \mid \sum z_i = 1, z_j \geq 0\}$. A homeomorphism from $\text{Conf}_n((0, 1))$ to the interior of $\Delta^n$ is given by the map

$$(x_1, \ldots, x_n) \mapsto (x_1, x_2 - x_1, x_3 - x_2, \ldots, x_n - x_{n-1}, 1 - x_n)$$

Since none of the $x_i$ are equal to each other, or to 0 or 1, the image consists of elements $(z_0, \ldots, z_n)$ where none of the $z_i$ are equal to 0 or 1; this is precisely the interior of the simplex.

The same map gives a homeomorphism of $\text{Sym}^n(\mathbb{R})$ onto the subspace of $\Delta^n$ consisting of points $(z_0, \ldots, z_n)$ where neither $z_0$ nor $z_n$ are equal to 0 or 1. We recall that $\Delta^n$ has $n+1$ faces, all of which are defined by a single coordinate equalling 0. If any of the $z_i$ are equal to 1, then all of the other $z_j$ are 0; this is a single vertex and contained in one of the faces. Thus $\text{Sym}^n(\mathbb{R})$ is the complement of two faces in $\Delta^n$.

In contrast, the symmetric product of $\mathbb{C}$ admits a much simpler description:

**Proposition 3.** $\text{Sym}^n(\mathbb{C})$ is homeomorphic to $\mathbb{C}^n$.

**Proof.** Let $\text{Poly}_n$ denote the space of monic, degree $n$ polynomials over $\mathbb{C}$;

$$\text{Poly}_n = \{f(z) = z^n + a_1z^{n-1} + \ldots + a_n z + a_0 \mid (a_0, \ldots, a_n) \neq 0\}$$

There is a homeomorphism $\text{Poly}_n \to \text{Sym}^n\mathbb{C}$ which carries $f$ to the unordered $n$-tuple of its (not necessarily distinct) roots. That this is a bijection is a consequence of the fundamental theorem of algebra.

Note that for an element $\hat{z} = (z_1, \ldots, z_n) \in \text{Sym}^n\mathbb{C}$, the monic polynomial $f$ with roots at $\hat{z}$ has coefficients $a_i = (-1)^ie_i(z_1, \ldots, z_n)$, where $e_i$ is the $i^{th}$ elementary symmetric polynomial. Thus an explicit set of coordinates on $\text{Sym}^n(\mathbb{C})$ is given by the elementary symmetric polynomials.

Finally, we have:

**Proposition 4.** $\text{Sym}^n(\mathbb{C}P^1)$ is homeomorphic to $\mathbb{C}P^n$.

**Proof.** Define

$$\text{Homog}_n := \{f(z, w) = a_0z^n + a_1z^{n-1}w + \cdots + a_{n-1}zw^{n-1} + a_nw^n \mid (a_0, \ldots, a_n) \neq 0\}$$

to be the space of nonzero homogenous polynomials of degree $n$ in two variables $z, w$; it is homeomorphic to $\mathbb{C}^{n+1} \setminus \{0\}$. Letting $\mathbb{C}^\times$ act by scaling the coefficients of such a polynomial, $\text{Homog}_n/\mathbb{C}^\times \cong \mathbb{C}P^n$. 

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Define a homeomorphism $\text{Sym}^n(\mathbb{C}P^1) \to \text{Homog}_n / \mathbb{C}^\times$ by

$$(z_1, \ldots, z_n) \mapsto f(z, w) := \prod_{i=1}^{n} (z - z_i w).$$

Here, if $z_i = \infty$, we interpret the factor $z - z_i w$ as $-w$. An inverse is given as follows: factor $f(z, w) = w^m g(z, w)$ for some $m$ so that $g(z, w)$ is indivisible by $w$. Then map $f$ to the $n$-tuple consisting of $m$ points at $\infty$, along with the $n - m$ roots in $\mathbb{C}$ of $g(z, 1)$ (which is of degree $n - m$).

The Fox–Neuwirth cell decomposition of $\text{Conf}_n(\mathbb{C})$

We will describe the results of [FN62] (see also [GS12]) which give a decomposition of $\text{Conf}_n(\mathbb{R}^m)$ into spaces homeomorphic to Euclidean spaces. This does not give a cell-decomposition of $\text{Conf}_n(\mathbb{R}^m)$, but rather of its 1-point compactification. We will restrict our focus to the case $m = 2$ (i.e., $\text{Conf}_n(\mathbb{C})$), and encourage the interested reader to extend these constructions to $m > 2$.

An ordered partition $\lambda = (\lambda_1, \ldots, \lambda_k)$ of $n$ has $\sum \lambda_i = n$. With apologies to the combinatorialists present, we will write $k = |\lambda|$ for the number of parts of $\lambda$. The $n^{\text{th}}$ symmetric product $\text{Sym}_n(\mathbb{R})$ of the real line has a stratification by these partitions:

$$\text{Sym}_n(\mathbb{R}) = \bigsqcup_{\lambda \vdash n} \text{Sym}_{\lambda}(\mathbb{R})$$

where elements of $\text{Sym}_{\lambda}(\mathbb{R})$ consist of $|\lambda|$ distinct points $x_1, \ldots, x_{|\lambda|}$, the $i^{\text{th}}$ of which has multiplicity $\lambda_i$. Further, since $\mathbb{R}$ is ordered, we insist that $x_1 < \cdots < x_{|\lambda|}$. This space is evidently homeomorphic to $\text{Conf}_{|\lambda|}(\mathbb{R})$, which is in turn homeomorphic to $\mathbb{R}^{|\lambda|}$, as shown above.

Define a map $\pi : \text{Conf}_n(\mathbb{C}) \to \text{Sym}_n(\mathbb{R})$ by taking real parts:

$$\pi(z_1, \ldots, z_n) = (\Re(z_1), \ldots, \Re(z_n)),$$

and let $\text{Conf}_\lambda(\mathbb{C})$ denote the preimage of $\text{Sym}_\lambda(\mathbb{R})$ under $\pi$. This subspace is homeomorphic to

$$\text{Sym}_\lambda(\mathbb{R}) \times \prod_{i=1}^{|\lambda|} \text{Conf}_{\lambda_i}(\mathbb{R}),$$

3This discussion is taken from a forthcoming paper which is joint with TriThang Tran; the results described are not original to us, although the presentation is. I suppose that I don’t feel too much shame in plagiarizing myself when summarizing someone else’s work.
where the configuration factors record the imaginary part of the configuration of $\lambda_i$ points lying over the $i^{th}$ term in the set of the real coordinates of $z$. We again employ the fact that $\text{Conf}_k(\mathbb{R}) \cong \mathbb{R}^k$ and conclude that $\text{Conf}_\lambda(\mathbb{C}) \cong \mathbb{R}^{n+|\lambda|}$.

**Proposition 5.** The collection of subspaces $\text{Conf}_\lambda(\mathbb{C})$ forms a cellular decomposition of the 1-point compactification $\text{Conf}_n(\mathbb{C}) \cup \{\infty\}$; the cells of dimension $d$ are indexed by those partitions $\lambda$ with $n + |\lambda| = d$. Furthermore, the closure of the cell $\text{Conf}_\lambda(\mathbb{C})$ is the union

$$\overline{\text{Conf}_\lambda(\mathbb{C})} = \coprod_{\rho} \text{Conf}_\rho(\mathbb{C})$$

over ordered partitions $\rho$ which are refined by $\lambda$.

We must explain the last comment. Loosely speaking, the boundaries of the cell described above occur in two ways. First, points in a configuration may approach each other or $\pm i\infty$ along vertical lines (in which case their boundary is the point at infinity). Secondly, the $i^{th}$ and $i + 1^{st}$ vertical columns of configurations may approach each other horizontally, in which case the associated component of the boundary is given in terms of the cell $\text{Conf}_\rho(\mathbb{C})$, where $\rho$ is obtained from $\lambda$ by summing $\lambda_i$ and $\lambda_{i+1}$.

It is worth noting that, other than $\{\infty\}$, there are no cells of dimension less than or equal to $n$ in this decomposition. Therefore, we have

**Corollary 6.** For $* \geq n$,

$$H_* \text{Conf}_n(\mathbb{C}) \cong H^{2n-*}_c \text{Conf}_n(\mathbb{C}) = H^{2n-*}(\text{Conf}_n(\mathbb{C}) \cup \{\infty\}, \{\infty\}) = 0.$$  

There are more computational proofs of this fact that rely on our previous computation of the homology of $P\text{Conf}_n(\mathbb{C})$, though perhaps none quite as enlightening. In any case, this follows from the cellular decomposition and the following exercise:

**Exercise 7.** $\text{Conf}_n(\mathbb{C})$ is a $2n$-dimensional, oriented, non-compact manifold.

A word of warning: there are $2^{n-1}$ ordered partitions of $n$, so this is not the most efficient approach to computing $H_* \text{Conf}_n(\mathbb{C})$ as $n$ grows.

**References**
