Week 5: Operads and iterated loop spaces

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Operads

For a thorough introduction to the basic language regarding operads, take a look at Peter May’s “Definitions: operads, algebras and modules” at http://math.uchicago.edu/~may/PAPERS/handout.pdf

In our language, the operad we called $\text{Assoc}$ is the operad $\mathcal{M}$ of Example 6, and the operad $\text{Com}$ is $\mathcal{N}$ of Example 7.

The little disks operads

For each positive integer $k$, a TD-map $f : D^k \to D^k$ is the composite of a translation and a (positive) dilation$^1$:

$$f(x) = ax + b, \ a \in \mathbb{R}_{>0}, \text{ and } b \in \mathbb{R}^k.$$  

**Definition 1.** The $k$-dimensional little disks operads $C_k$ is defined by the spaces

$$C_k(n) := \{(f_1, \ldots, f_n) \mid f_i \text{ are TD-maps, and } f_i \text{Int}(D^k) \cap f_j \text{Int}(D^k) = \emptyset \text{ if } i \neq j\}.$$  

We may topologize $C_k(n)$ as a subspace of $(\mathbb{R}_{>0} \times \mathbb{R}^k)^n$ via the map which carries $(f_i)_{i=1}^n$ to $(a_i, b_i)_{i=1}^n$ The action of $S_n$ on $C_k(n)$ is by permutation of the $f_i$.

To define the operad composition $\gamma$, we note that an element $(f_1, \ldots, f_n) \in C_k(n)$ defines a function $F : \bigsqcup_n D^k \to D^k$ whose value on the $i$th copy of $D^k$ is given by $f_i$. By assumption, this map is injective on the interior of the domain. For elements $G^{(i)} = (g^{(i)}_1, \ldots, g^{(i)}_j) \in C_k(j_i)$, we may define

$$\gamma(F ; G^{(1)}, \ldots, G^{(n)}) = F \circ (G^{(1)} \sqcup \cdots \sqcup G^{(n)}) : \bigsqcup_{\sum j_i} D^k \to D^k.$$  

$^1$If we additionally allow rotations, we obtain the notion of a TDR-map; the associated operad is called the framed little disks operad.
The unit of the operad is the identity map \( \text{id}_{\mathcal{O}} \in \mathcal{O}(1) \). We leave it to the reader to verify that this does indeed define an operad of topological spaces.

Consider the map \( p: \mathcal{C}_k(n) \to \text{PConf}_n(D^k) \) which carries a collection of little \( k \)-disks to their centers; that is, \( p([f_i]_{i=1}^n) = (b_i)_{i=1}^n \).

**Proposition 2.** The map \( p: \mathcal{C}_k(n) \to \text{PConf}_n(D^k) \) is a homotopy equivalence.

**Proof.** For a configuration \( \bar{x} \in \text{PConf}_n(D^k) \), define

\[
d(\bar{x}) = \inf \left\{ \frac{1}{2}|x_i - x_j|, i \neq j; |x_i - z|, z \in \partial D^k \right\}
\]

Then \( d: \text{PConf}_n(D^k) \to \mathbb{R}_{>0} \) is a continuous function. Define \( s: \text{PConf}_n(D^k) \to \mathcal{C}_k(n) \) by \( s(\bar{x}) = (f_1, \ldots, f_n) \), where

\[
f_i(z) = d(\bar{x})z + x_i.
\]

It is immediate that \( p \circ s = \text{id} \). In the other direction, if \( f_i(z) = a_i z + b_i \), the \( i \)-th component of \( s(p(f_1, \ldots, f_n)) \) is the map \( z \mapsto d(\bar{b})z + b_i \). A homotopy \( s \circ p \simeq \text{id} \) on \( \mathcal{C}_k(n) \) is given by the straightline homotopy from \( a_i \) to \( d(\bar{b}) \).

\[\square\]

Define a map \( q: \mathcal{C}_1(n) \to \text{Assoc}(n) \) which carries a configuration \( (f_1, \ldots, f_n) \) of little disks in \( D^1 \) to the unique permutation \( \sigma \in S_n \) with the property that \( b_{\sigma(1)} < \cdots < b_{\sigma(n)} \). It is not hard to show that \( q \) is a map of operads.

**Corollary 3.** The map \( q: \mathcal{C}_1 \to \text{Assoc} \) is a homotopy equivalence of operads.

**Proof.** We have shown that \( \mathcal{C}_1(n) \simeq \text{PConf}_n(D^1) \). In the previous lecture, we showed\(^2\) that \( \text{PConf}_n(D^1) \) is homeomorphic to \( \text{Int}(\Delta^n) \times S_n \). The projection map from this space onto \( S_n = \text{Assoc}(n) \) is precisely \( q \), and is clearly a homotopy equivalence.

\[\square\]

It is worth mentioning that while \( q \) has a homotopy inverse in the category of (sequences of) spaces, there is no homotopy inverse which is a map of operads. If \( A \) is an algebra for \( \text{Assoc} \) (that is, a strictly associative, unital H-space), then the composite

\[
q: \mathcal{C}_1 \to \text{Assoc} \to \text{End}_A
\]

makes \( A \) an algebra for \( \mathcal{C}_1 \). On the other hand, the non-existence of an operadic section of \( q \) ensures that there are \( \mathcal{C}_1 \)-algebras that are not strictly associative H-spaces. However, we will see in the future that every \( \mathcal{C}_1 \)-algebra is homotopy equivalent to an \( \text{Assoc} \)-algebra.

\(^2\)Or rather, we showed the corresponding statement for the unordered configuration space; the same proof holds here.
The operadic point of view underscores a general philosophy in “homotopy coherent mathematics”: equalities between operations often must be replaced by homotopies. However, homotopies themselves are usually too unstructured a notion to be of much use. If, however, those homotopies are encoded as paths in a structured object like an operad, a more useful\(^3\) notion arises.

Example 4. The projection map \( p : C_k \rightarrow \text{Com} \) which collapses everything in \( C_k(n) \) to a point is trivially a map of operads. For \( k > 1 \), it is an isomorphism in \( \pi_0 \). Consequently, every \( C_k \)-algebra has a homotopy commutative multiplication. This may be seen explicitly as follows for \( k = 2 \): an arbitrary element \( \mu \in C_2(2) \) defines a multiplication on a \( C - 2 \)-algebra \( A \). A Dehn twist around the midpoint of the centers of the little disks in \( \mu \) gives a path from \( \mu \) to \( \mu \circ \sigma \), where \( \sigma = (12) \) is the permutation swapping the two inputs. This yields a homotopy between \( a \ast b = \mu(a, b) \) and \( b \ast a = \mu \circ \sigma(a, b) \).

The recognition theorem

The recognition theorem asserts that connected\(^4\) \( C_k \)-algebras and \( k \)-fold loop spaces are essentially the same thing. Half of this is straightforward.

Proposition 5. For any based space \((X, \ast)\) and integer \( k \geq 0 \), \( \Omega^k X = \text{Map}((D^k, \partial), (X, \ast)) \) is a \( C_k \)-algebra.

Proof. Let \( g_1, \ldots, g_n \in \Omega^k X \) and \( f = (f_1, \ldots, f_n) \in C_k(n) \) (so that the \( f_i \) are TD-maps). Define the action of \( C_k \) on \( \Omega^k X \) as follows: for \( d \in D^k \), the value of \( \theta(f; (g_1, \ldots, g_n)) \) on \( d \) is

\[ \theta(f; (g_1, \ldots, g_n))(d) = \begin{cases} \ast, & d \notin \text{im}(f_i) \text{ for any } i. \\ g_i f_i^{-1}(d), & d \in \text{im}(f_i). \end{cases} \]

An enthralling computation verifies that this does indeed satisfy the axioms of an operad action.

This has a sort of converse, which is much less straightforward, and will require some substantial work:

Theorem 6 (Recognition, [May72]). If \( Y \) is a connected \( C_k \)-algebra, there exists a space \( X \) and a weak equivalence of \( C_k \)-algebras \( \Omega^k Y \simeq X \).

\(^3\)For instance, a homotopy associative, homotopy unital H-space does not have a well-structured category of modules, whereas an algebra over \( C_1 \) does.

\(^4\)In the disconnected case, look forward to the group completion theorem in the near future.
Free algebras and the approximation theorem

Definition 7. Let \((Z, *)\) be a pointed topological space and \(k > 0\) an integer. The free \(C_k\)-algebra on \((Z, *)\) is the quotient space

\[
C_k[Z] := \left( \prod_{n \geq 0} C_k(n) \times_{S_n} Z^{\times n} \right) / \sim
\]

where \(((f_1, \ldots, f_n), (z_1, \ldots, z_{n-1})) \sim ((f_1, \ldots, f_{n-1}), (z_1, \ldots, z_{n-1}))\).

One thinks of this as the space of little \(k\)-disks in the unit \(k\)-disk, decorated by points in \(Z\). Further, when a little disk is decorated by the basepoint of \(Z\), one may drop it and its label. This is in fact a \(C_k\)-algebra: the algebra structure is induced by the operadic composition in \(C_k\) (the decorations get carried along for the ride). In fact, \(C_k[Z]\) is the free \(C_k\)-algebra on \(Z\) in the following sense:

Proposition 8. There exists a natural bijection

\[
\text{Map}_{C_k-\text{alg}}(C_k[Z], W) = \text{Map}_{\text{Top}^*}(Z, W).
\]

Proof. The map from left to right restricts a \(C_k\)-algebra map \(C_k[Z] \to W\) to the subspace

\[
Z = \{\text{id}\} \times Z \subseteq C_k(1) \times Z \subseteq C_k[Z].
\]

In the other direction, for \(h : Z \to W\) an arbitrary continuous, pointed map, define \(H : C_k[Z] \to W\) by \(H(f, (z_1, \ldots, z_n)) = \theta(f; (h(z_1), \ldots, h(z_n)))\).

Define \(e : C_k[Z] \to \Omega^k\Sigma^k Z\) as the free map generated by \(Z \to \Omega^k\Sigma^k Z\) which is adjoint to the identity on \(\Sigma^k Z\). More prosaically, \(e\) carries \(((f_1, \ldots, f_n), (z_1, \ldots, z_n))\) to the map \(S^k \to \Sigma^k Z\) which wraps the little disk \(f_i \subseteq D^k\) around \(S^k = \Sigma^k\{*, z_i\} \subseteq \Sigma^k Z\), and carries the rest of \(S^k = D^k/\partial\) to the basepoint *.

Theorem 9 (Approximation [May72]). If \(Z\) is connected, then \(e\) is a weak equivalence.

We will prove this result in the next lecture. It is needed to prove the recognition theorem. Historically, this result was used to investigate the homology of iterated loop spaces, since \(C_k[Z]\) is more geometrically tractable than \(\Omega^k\Sigma^k Z\). One may also reverse the flow of information: from knowledge of the homology of function spaces, we may come to an understanding of the homology of (decorated) configuration spaces.

References