The relaxed game chromatic index of trees

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A graph $G = (V, E)$: A tree $T = (V, E)$ (no cycles):

![Tree illustration]

Not a tree ($\geq 1$ cycle):

![Not a tree illustration]

Degree of a vertex $v$: the number of vertices adjacent to $v$.

$\Delta(T)$ = maximum degree of a tree $T$. 
A **proper edge coloring** of a graph $G$ is an assignment of colors to each edge of $G$ such that if $e$ and $f$ are adjacent edges (i.e., share a common vertex), then $e$ and $f$ are assigned different colors. For example:

![Proper and improper colorings](image)

**Figure:** Proper and improper colorings (left to right).

The minimum number of colors needed to properly color the edges of a graph $G$ is called the **chromatic index** of $G$, denoted $\chi'(G)$.
It’s not difficult to see that \( \Delta(G) \leq \chi'(G) \) for any graph \( G \).

Less trivially, we have the following upper bound for the chromatic index:

**Theorem (Vizing, 1964)**

For any graph \( G \), \( \chi'(G) \leq \Delta(G) + 1 \).

Thus, \( \Delta(G) \leq \chi'(G) \leq \Delta(G) + 1 \) for any graph \( G \).

Unfortunately, it’s NP-complete to determine whether the chromatic index of a graph is \( \Delta \) or \( \Delta + 1 \) (Holyer, 1981).
Relaxed edge colorings

A \textit{d-relaxed edge coloring} is an edge coloring in which we allow an edge \( e \) to be colored \( \alpha \) if:

- \( e \) is adjacent to at most \( d \alpha \)-colored edges, and
- if \( f \) is an \( \alpha \)-colored edge adjacent to \( e \), then \( f \) is adjacent to at most \( d - 1 \alpha \)-colored edges.

For example:

In a 2-relaxed edge coloring, \( e \) could be colored with blue, but \textit{not} red.
The relaxed edge coloring game

The \((r, d)\)-relaxed edge coloring game on a graph \(G\) is as follows:

- A two player game—say Alice and Bob, where Alice plays first.
- Alice and Bob alternate coloring one uncolored edge per turn.
- \(r\) colors available.
- The players agree to a \(d\)-relaxed proper coloring.
- If the graph is properly colored, Alice wins.
- In the case of an uncolorable edge, Bob wins.
For a fixed defect $d$, the $d$-relaxed game chromatic index of a graph $G$, denoted $^d\chi'_g(G)$, is the least number of colors $r$ such that Alice has a winning strategy in the $(r, d)$-relaxed edge coloring game.

We write the 0-relaxed edge-game chromatic index as $\chi'_g(G)$.

For a fixed number of colors $r$, the $r$-edge-game defect of a graph $G$, denoted $\text{def}'_g(G, r)$, is the least defect $d$ such that Alice has a winning strategy in the game.
What’s already known . . .

**Theorem (Cai & Zhu, 2001)**

*For any tree $T$, $\chi'_g(T) \leq \Delta(T) + 2$.*

**Theorem (Dunn, 2007)**

*For any tree $T$ with $\Delta(T) = \Delta$, $\text{def}'_g(T, \Delta + 1) \leq 1$. Furthermore, if $d \geq 1$, then $d^{\chi'_g(T)} \leq \Delta + 1$.  

**Theorem (Dunn, 2007)**

*For any tree $T$ with $\Delta(T) = \Delta$, $\text{def}'_g(T, \Delta) \leq 3$. Furthermore, if $d \geq 3$, then $d^{\chi'_g(T)} \leq \Delta$.  

Can we keep going? . . . yes.
The main result

Theorem (Dunn, M., Nordstrom, 2009)

Let $T$ be a tree with $\Delta(T) = \Delta$. Then, for any $k = 1, 2, \ldots, \Delta - 1$, 
$
\text{def}_g'(T, \Delta - k) \leq 2k + 2. 
$
Furthermore, if $d \geq 2k + 2$, then $d\chi'_g(T) \leq \Delta - k$. 

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For a colored edge $e$, we let $c(e)$ denote the color of $e$.

For an edge $e$, the **defect** of $e$, denoted $\text{def}(e)$, is the number of edges adjacent to $e$ having the same color as $e$. If $e$ is uncolored, $\text{def}(e) = 0$.

A color $\alpha$ is **eligible** for an edge $e$ if the parent of $e$ is not colored $\alpha$.

**Alice only uses eligible colors.**
We fix a tree $T = (V, E)$ with $\Delta(T) = \Delta$.

We also fix $k = 1, 2, \ldots, \Delta - 1$.

Recall, our result tells us a sufficient defect when Alice and Bob are playing the game with $\Delta - k$ colors.
Definition: secure

For an edge $e$, we say that $B[e]$ is secure if there exists a colored $Y \subseteq B[e]$ such that $|Y| \geq k$ and, for every edge $f \in Y$, there exists $f' \in B[e] - Y$ with $c(f) = c(f')$.

For example:

![Diagram](image)

Here we have $\Delta = 7$ and $k = 3$. If Alice and Bob are playing with $\Delta - k = 4$ colors, then there is an eligible color for $e$ that does not appear among the siblings of $e$.

This is always the case when $B[e]$ is secure, for any edge $e$. 
Alice’s strategy consists of a

- **Search stage:** Alice locates the edge $e$ that she will color, in response to the move just made by Bob.

- **Coloring stage:** Alice chooses an eligible color for $e$, with the hope of minimizing the defect of nearby edges.
Alice’s strategy: activation

Alice will maintain a set $A \subseteq E$ of active edges.

Once an edge is activated—i.e., put into $A$—it remains active for the remainder of the game.

All colored edges are active.

Alice activates edges in response to moves made by Bob.
Alice’s strategy: searching

Suppose Bob has just colored an edge $b$ the same color as its parent $p(b)$.

1. If the grandparent $p^2(b)$ is uncolored, Alice selects it and moves to the coloring stage.

2. Otherwise, Alice selects any uncolored sibling of $p(b)$.
Alice’s strategy: coloring

Suppose Alice is about to color an edge $e$:

- If $B[e]$ is secure, then Alice chooses an eligible color for $e$ that does not appear among the siblings of $e$.
- If possible, let $f$ be the last edge to be colored with an eligible color for $e$, such that $p(f)$ is a sibling of $e$ and $c(f) = c(p(f))$. If such an edge exists, Alice colors $e$ with $c(f)$.
- Otherwise, Alice chooses an eligible color for $e$ that minimizes $\text{def}(e)$.
Given any tree $T$ with $\Delta(T) = \Delta$, $k = 1, 2, \ldots, \Delta - 1$, and considering the relaxed edge coloring game with $\Delta - k$ colors:

We have outlined a strategy for Alice such that she may always make a legal move, while keeping the defect of any edge at most $2k + 2$.

Furthermore, if the defect is greater than $2k + 2$, the arguments remain valid.

Since Bob may adopt Alice’s strategy at any point during the game, both players always have a legal move that respects the defect. Thus,

- $\text{def}_g'(T, \Delta - k) \leq 2k + 2$ and,
- $\text{g}_g'(T) \leq \Delta - k$, whenever $d \geq 2k + 2$. 


Thanks!

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