Uniform distribution: approximating continuous objects by discrete ones

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“Introduction to Research” seminar
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A sequence \((x_n)\) is *uniformly distributed* in \([0, 1]\) iff

for any interval \(I \subset [0, 1]\) : \(\lim_{N \to \infty} \frac{\#\{n \leq N : x_n \in I\}}{N} = |I|\)
Uniform distribution of sequences

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- Equivalently, for all continuous \(f\) on \([0, 1]\):

\[
\frac{1}{N} \sum_{n=1}^{N} f(x_n) \longrightarrow \int_{0}^{1} f(x) \, dx \text{ as } N \to \infty.
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**Weyl Criterion (1916):**

\((x_n)\) is uniformly distributed in \([0, 1]\) iff for all \(k \in \mathbb{Z}\), \(k \neq 0\):

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i k x_n} = 0
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The sequence \(\{n\theta\}\) is uniformly distributed in \([0, 1]\) iff \(\theta\) is irrational.
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The sequence \(\{n\theta\}\) is uniformly distributed in \([0, 1]\) iff \(\theta\) is irrational.

For any subsequence \((n_k)\) of integers, the sequence \(\{n_k\theta\}\) is uniformly distributed for a.e. \(\theta\).
Discrepancy of a sequence

For a sequence $\omega = (\omega_n)_{n=1}^{\infty}$ and an interval $I \subset [0, 1]$ consider the quantity

$$\Delta_{N,I} = \# \{ \omega_n : \omega_n \in I; n \leq N \} - N|I|.$$ 

Define

$$D_N = \sup_{I \subset [0,1]} |\Delta_{N,I}|.$$
Discrepancy of a sequence

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Define

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$$

A sequence $(\omega_n)_{n=1}^{\infty}$ is u.d. in $[0, 1]$ if and only if

$$
\lim_{N \to \infty} \frac{D_N}{N} = 0.
$$
Theorem (Erdős-Turan)

For any sequence $\omega \subset [0, 1]$ we have

$$D_N(\omega) \lesssim \frac{N}{m} + \sum_{h=1}^{m} \frac{1}{h} \left| \sum_{n=1}^{N} e^{2\pi i h \omega n} \right|$$

for all natural numbers $m$. 

Folklore: misses optimal estimates by a logarithm. E.g., for a badly approximable irrational $\theta$, Erdős-Turan yields $D_N(\{n\theta\}) \lesssim \log 2 N$, while in fact $D_N(\{n\theta\}) \lesssim \log N$.

For $\omega = \{n\theta\}$ sharper bounds can be obtained using continued fractions.
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Erdős-Turan inequality

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Can discrepancy stay small?

Consider a sequence \( \omega = (\omega_n)_{n=1}^{\infty} \subset [0, 1] \).
Can discrepancy stay small?

Consider a sequence $\omega = (\omega_n)_{n=1}^\infty \subset [0, 1]$.

van der Corput (1934): Can $D_N(\omega)$ be bounded as $N \to \infty$?

van Aardenne-Ehrenfest (1945): NO!

Theorem (K. Roth, 1954)
The following are equivalent:

(i) For every $\omega = (\omega_n)_{n=1}^\infty \subset [0, 1]$, $D_N(\omega) \gtrsim f(N)$ for infinitely many values of $N$.

(ii) For any distribution $P_N \subset [0, 1]^2$ of $N$ points,

\[
\sup_{R - rectangle} \left| \# P_N \cap R - N \cdot |R| \right| \gtrsim f(N)
\]
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Irregularities of distribution

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(i) For every $\omega = (\omega_n)_{n=1}^{\infty} \subset [0, 1]$,

$$D_N(\omega) \gtrsim f(N)$$

for infinitely many values of $N$.

(ii) For any distribution $\mathcal{P}_N \subset [0, 1]^2$ of $N$ points,

$$\sup_{R-\text{rectangle}} \left| \# \mathcal{P}_N \cap R - N \cdot |R| \right| \gtrsim f(N)$$
Irregularities of Distribution: simplest example

\[ X - \text{roll of a single die} \]
Irregularities of Distribution: simplest example

$X$ – roll of a single die

$\left| X - \mathbb{E}X \right| \geq \frac{1}{2}$
Geometric Discrepancy

$\mathcal{P}_N$ – a set of $N$ points in $[0, 1]^d$

$\mathcal{R}$ – a geometric family (e.g. axis-parallel rectangles, all rectangles, polytopes, balls, convex sets etc.)
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Discrepancy of \( \mathcal{P}_N \) with respect to \( R \in \mathcal{R} \)

\[
D(\mathcal{P}_N, R) = \#\{\mathcal{P}_N \cap R\} - N \cdot \text{vol}(R)
\]
Geometric Discrepancy

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$$D(\mathcal{P}_N, R) = \#\{\mathcal{P}_N \cap R\} - N \cdot \text{vol}(R)$$

Discrepancy of $\mathcal{P}_N$ with respect to $\mathcal{R}$

$$D(\mathcal{P}_N) = \sup_{R \in \mathcal{R}} |D(\mathcal{P}_N, R)|$$
Geometric Discrepancy

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D(\mathcal{P}_N) = \sup_{R \in \mathcal{R}} |D(\mathcal{P}_N, R)|
\]

Discrepancy of \( \mathcal{P}_N \) with respect to \( \mathcal{R} \)

\[
D(N) = \inf_{\mathcal{P}_N} D(\mathcal{P}_N)
\]
Consider a set $\mathcal{P}_N \subset [0, 1]^d$ consisting of $N$ points:

$$D_N(x) = \#\{\mathcal{P}_N \cap [0, x]\} - N x_1 x_2 \ldots x_d$$
Koksma-Hlawka inequality:

\[
\left| \int_{[0,1]^d} f(x) \, dx - \frac{1}{N} \sum_{p \in \mathcal{P}_N} f(p) \right| \lesssim \frac{1}{N} V(f) \cdot \|D_N\|_{\infty}
\]
Koksma-Hlawka inequality:

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- $V(f)$ is the Hardy-Krause variation of $f$
Koksma-Hlawka inequality:

\[ \left| \int_{[0,1]^d} f(x) \, dx - \frac{1}{N} \sum_{p \in \mathcal{P}_N} f(p) \right| \lesssim \frac{1}{N} \| f_{x_1 \ldots x_d} \|_1 \cdot \| D_N \|_\infty \]

- \( V(f) \) is the Hardy-Krause variation of \( f \)
- \( V(f) = \int_{[0,1]^d} \left| \frac{\partial^d f}{\partial x_1 \partial x_2 \ldots \partial x_d} \right| \, dx_1 \ldots dx_d \)
  e.g., if \( f(x_1, \ldots, x_d) = \int_{x_1}^1 \ldots \int_{x_d}^1 \phi(y) \, dy \)
Theorem (ROTH, K. F. On irregularities of distribution, Mathematika 1 (1954), 73–79.)

There exists $C_d \geq 0$ such that for any $N$-point set $\mathcal{P}_N \subset [0, 1]^d$

$$\|D_N\|_2 \geq C_d (\log N)^{\frac{d-1}{2}}.$$
According to Roth himself, this was his favorite result.

- William Chen (private communication)
- Kenneth Stolarsky (private communication)
- Ben Green (comment on Terry Tao’s blog)

12 comments

12 November, 2015 at 9:55 am
Ben Green

I did meet Roth, in Inverness around 7 years ago. I asked him what his favourite proof (among his results was) and he said the lower bound for the L^2 discrepancy of point sets with respect to axis parallel boxes. It is a very elegant argument, nicely described in Bernard Chazelle’s book “Discrepancy Theory”. Later in his career he became quite interested in the “Heilbronn triangle problem”, which came up in conversation the other day: given n points in the unit square, what’s the smallest area of triangle they are guaranteed to span. I believe that n^{-2+o(1)} is conjectured, and that Roth was the first to improve on the trivial bound O(1/n). Subsequently bounds of the form O(n^{-1-c}) were obtained.

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- 4 papers by Roth (On irregularities of distribution. I–IV)
- 10 papers by W.M. Schmidt (On irregularities of distribution. I–X)
- 2 by J. Beck (Note on irregularities of distribution. I–II)
- 4 by W. W. L. Chen (On irregularities of distribution. I–IV)
- 2 by Beck and Chen (Note on irregularities of distribution. I–II)
- a book by Beck and Chen, “Irregularities of distribution”. 
References

Books

- Kuypers, Niederreiter
  “Uniform distribution of sequences”
- Beck, Chen
  “Irregularities of distribution”
- Drmota, Tichy
  “Sequences, discrepancies and applications”
- Matoušek
  “Geometric discrepancy”
- Dick, Pillichshammer
  “Digital nets and sequences”
- Chazelle
  “Discrepancy method”
Theorem (Roth, 1954 ($p = 2$); Schmidt, 1977 ($1 < p < 2$))

The following estimate holds for all $\mathcal{P}_N \subset [0, 1]^d$ with $\#\mathcal{P}_N = N$:

$$\|D_N\|_p \gtrsim (\log N)^{\frac{d-1}{2}}$$
Theorem (Roth, 1954 ($p = 2$); Schmidt, 1977 ($1 < p < 2$))

The following estimate holds for all $\mathcal{P}_N \subset [0, 1]^d$ with $\#\mathcal{P}_N = N$:

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Theorem (Davenport, 1956 ($d = 2$, $p = 2$); Roth, 1979 ($d \geq 3$, $p = 2$); Frolov, 1980 ($p > 2$, $d = 2$); Chen, 1983 ($p > 2$, $d \geq 3$); Chen, Skriganov, 2000’s)

There exist sets $\mathcal{P}_N \subset [0, 1]^d$ with

$$\|D_N\|_p \lesssim (\log N)^{d-1/2}$$
Conjecture

\[ \| D_N \|_\infty \gg (\log N)^{\frac{d-1}{2}} \]
Conjecture

$$\|D_N\|_\infty \gg (\log N)^{\frac{d-1}{2}}$$

Theorem (Schmidt, 1972; Halász, 1981)

In dimension $d = 2$ we have $\|D_N\|_\infty \gtrsim \log N$
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Theorem (Schmidt, 1972; Halász, 1981)

In dimension \( d = 2 \) we have \( \|D_N\|_\infty \gtrsim \log N \)

\( d = 2 \): Lerch, 1904; van der Corput, 1934

There exist \( \mathcal{P}_N \subset [0, 1]^2 \) with \( \|D_N\|_\infty \approx \log N \)
Low discrepancy sets

The irrational ($\alpha = \sqrt{2}$) lattice with $N = 2^{12}$ points

$(n/N, \{n\alpha\}), \quad n = 0, 1, \ldots, N - 1.$

Discrepancy $\approx \log N$
Low discrepancy sets

The van der Corput set with $N = 2^n$ points (here $n = 12$)

$$(0.x_1x_2...x_n, 0.x_nx_{n-1}...x_2x_1), \quad x_k = 0 \text{ or } 1.$$

Discrepancy $\approx \log N$
van der Corput set

van der Corput set with $N = 2^3$ points
van der Corput set with $N = 2^4$ points
van der Corput set with $N = 2^5$ points
van der Corput set with $N = 2^6$ points
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van der Corput set

van der Corput set with $N = 2^8$ points
van der Corput set with $N = 2^9$ points
van der Corput set with $N = 2^{10}$ points
van der Corput set

van der Corput set with $N = 2^{11}$ points
van der Corput set

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van der Corput set
Conjecture
\[ \| D_N \|_\infty \gg (\log N)^{d-1/2} \]

Theorem (Schmidt, 1972; Halász, 1981)

_In dimension \( d = 2 \) we have_ \[ \| D_N \|_\infty \gtrsim \log N \]

\( d = 2 \): Lerch, 1904; van der Corput, 1934

_There exist_ \( \mathcal{P}_N \subset [0,1]^2 \) _with_ \[ \| D_N \|_\infty \approx \log N \]
$L^\infty$ estimates

**Conjecture**

\[ \|D_N\|_\infty \gg (\log N)^{\frac{d-1}{2}} \]

**Theorem (Schmidt, 1972; Halász, 1981)**

*In dimension \(d = 2\) we have* \(\|D_N\|_\infty \gtrsim \log N\)

**d = 2:** Lerch, 1904; van der Corput, 1934

There exist \(\mathcal{P}_N \subset [0, 1]^2\) with \(\|D_N\|_\infty \approx \log N\)

**d \geq 3,** Halton, Hammersley (1960):

There exist \(\mathcal{P}_N \subset [0, 1]^d\) with \(\|D_N\|_\infty \lesssim (\log N)^{d-1}\)
Conjectures and results

Conjecture 1

\[ \| D_N \|_\infty \gtrsim (\log N)^{d-1} \]
Conjectures and results

Conjecture 2

\[ \|D_N\|_\infty \gtrsim (\log N)^{\frac{d}{2}} \]
Conjectures and results

Conjecture 2
\[ \| D_N \|_\infty \gtrsim (\log N)^{\frac{d}{2}} \]

Theorem (DB, M.Lacey, A.Vagharshakyan, 2008)
For \( d \geq 3 \) there exists \( \eta > 0 \) such that the following estimate holds for all \( N \)-point distributions \( \mathcal{P}_N \subset [0, 1]^d \):
\[ \| D_N \|_\infty \gtrsim (\log N)^{\frac{d-1}{2} + \eta}. \]
Connections between problems

Discrepancy estimates
\[ \|D_N\|_\infty \gtrsim (\log N)^{\frac{d}{2}} \]

Small Ball Conjecture
\[ n^{d-2} \left\| \sum_{|R|=2^{-n}} \alpha_R h_R \right\|_\infty \gtrsim 2^{-n} \sum_{R: |R|=2^{-n}} |\alpha_R|. \]

Small deviations for the Brownian sheet
\[ -\log P(\|B\|_{C([0,1]^d)} < \varepsilon) \approx \varepsilon^{-2} (\log 1/\varepsilon)^{2d-1} \]

Metric entropy of \( MW^2 \)
\[ \log N(\varepsilon, 2, d) \approx \varepsilon^{-1} (\log 1/\varepsilon)^{d-1/2} \]

Kuelbs, Li, ’93
Lower and upper bounds in dimension $d = 2$

<table>
<thead>
<tr>
<th>LOWER BOUND</th>
<th>UPPER BOUND</th>
</tr>
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<tbody>
<tr>
<td><strong>Axis-parallel rectangles</strong></td>
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<tr>
<td>$D(N, A)$</td>
<td>$\log N$</td>
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<tr>
<td>$D_2(N, A)$</td>
<td>$\sqrt{\log N}$</td>
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<td>$N^{1/4}$</td>
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<td><strong>Convex Sets</strong></td>
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Geometric discrepancy

- No rotations: discrepancy $\approx \log N$

- All rotations: discrepancy $\approx N^{1/4}$
  (J. Beck, H. Montgomery)

- Partial rotations
  (lacunary sets, sets of small Minkowski dimension, etc)
  DB, X.Ma, C. Spencer, J. Pipher (2009-2011)
Higher dimensions: $d \geq 3$

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<td>$(\log N)^{\frac{d-1}{2}} + \eta$</td>
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Transference: geometric to combinatorial discrepancy

$S$ – a set with $N$ elements, $\mathcal{H}$ – a collection of subset of $S$,
$\chi : S \to \{-1, 1\}$ – 2-coloring (red-blue)

Combinatorial discrepancy: $\text{disc}(\mathcal{H}) = \min_{\chi} \max_{A \in \mathcal{H}} \left| \sum_{x \in A} \chi(x) \right|$
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\]

Combinatorial discrepancy generated by geometric systems:
Let \( \mathcal{A} \) be a family of measurable sets and \( S_N \) a set of \( N \) points.
\[
\text{disc}(S_N, \mathcal{A}) = \text{disc}(\{ S_N \cap A : A \in \mathcal{A} \})
\]
\[
\text{disc}(N, \mathcal{A}) = \sup_{S_N \subset [0,1]^d; \#S_N=N} \text{disc}(S_N, \mathcal{A})
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$$\text{disc}(S_N, A) = \text{disc}(\{S_N \cap A : A \in A\})$$
$$\text{disc}(N, A) = \sup_{S_N \subset [0,1]^d; \#S_N=N} \text{disc}(S_N, A)$$

Lemma (Sós; Beck; Lovász, Spencer, Vesztergombi; ...)

Combinatorial discrepancy “is larger than” the geometric discrepancy

$$D(N, A) \ll \text{disc}(N, A).$$
Let $R_d = \{\text{axis-parallel rectangles}\}$.

Tusnády’s problem:
What is the asymptotics of $T(N) = \text{disc}(N, R_d)$ as $N \to \infty$?

- $d = 2$: Matoušek; Beck
  \[ \log N \lesssim T(N) \lesssim \log^{5/2} N \]

- $d \geq 3$: Nikolov, Matoušek, 2014; Beck
  \[ (\log N)^{d-1} \lesssim T(N) \lesssim (\log N)^{d+\frac{1}{2}} \]
For $x \in S^d \subset \mathbb{R}^{d+1}$, $t \in [-1, 1]$ define spherical caps:

$$C(x, t) = \{ y \in S^d : \langle x, y \rangle \geq t \}.$$

For a finite set $Z = \{z_1, z_2, ..., z_N\} \subset S^d$ define

$$D_{\text{cap}}(Z) = \sup_{x \in S^d, t \in [-1, 1]} \left| \frac{\#(Z \cap C(x, t))}{N} - \sigma(C(x, t)) \right|.$$

**Theorem (Beck)**

*There exists an $N$-point set $Z \subset S^d$ with*

$$D_{\text{cap}}(Z) \lesssim N^{-\frac{1}{2} - \frac{1}{2d}} \sqrt{\log N}.$$

**Theorem (Beck)**

*For any $N$-point set $Z \subset S^d$*

$$D_{\text{cap}}(Z) \gtrsim N^{-\frac{1}{2} - \frac{1}{2d}}.$$
Define the spherical cap $L^2$ discrepancy

\[
D_{\text{cap}}^{(2)} = \left( \int_{S^{d-1}} \int_{-1}^{1} \left| \frac{\#(Z \cap C(x, t))}{N} - \sigma(C(x, t)) \right|^2 \, dt \, d\sigma(x) \right)^{\frac{1}{2}}.
\]
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**Theorem (Stolarsky invariance principle)**

*For any finite set $Z = \{z_1, \ldots, z_N\} \subset S^{d-1}$*

$$\frac{1}{N^2} \sum_{i,j=1}^{N} \|z_i - z_j\| + c_d \left[ D_{\text{cap}}^{(2)} \right]^2 = \text{const}$$

$$= \int_{S^{d-1}} \int_{S^{d-1}} \|x - y\| \, d\sigma(x) \, d\sigma(y).$$
Let \( x, y \in \mathbb{S}^d \) and choose a random hyperplane \( z^\perp \), where \( z \in \mathbb{S}^d \).
Let \( x, y \in \mathbb{S}^d \) and choose a random hyperplane \( z \perp \), where \( z \in \mathbb{S}^d \).

Then

\[
\mathbb{P}(z \perp \text{ separates } x \text{ and } y) = \mathbb{P}(\text{sign} \langle z, x \rangle \neq \text{sign} \langle z, y \rangle) = d(x, y),
\]

where \( d \) is the normalized geodesic distance on the sphere, i.e.

\[
d(x, y) = \frac{\cos^{-1} \langle x, y \rangle}{\pi}.
\]
Consider a finite set of vectors $Z = \{z_1, z_2, ..., z_N\}$ on the sphere $S^d$. Define the Hamming distance as

$$d_H(x, y) := \frac{\#\{z_k \in Z : \text{sgn}(x \cdot z_k) \neq \text{sgn}(y \cdot z_k)\}}{N},$$

i.e. the proportion of hyperplanes $z_k \perp$ that separate $x$ and $y$. 
Define

$$\Delta_Z(x, y) := d_H(x, y) - d(x, y).$$
Uniform tessellations

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Let $K \subset S^d$. We say that $Z$ is a $\delta$-uniform tessellation of $K$ if

$$\sup_{x, y \in K} |\Delta_Z(x, y)| \leq \delta.$$
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We say that \( Z \) is a \( \delta \)-uniform tessellation of \( K \) if
\[ \sup_{x, y \in K} |\Delta_Z(x, y)| \leq \delta. \]

Question:
Given \( K \subset \mathbb{S}^d \) and \( \delta > 0 \), what is the smallest value of \( N \) so that there exist a \( \delta \)-uniform tessellation of \( K \) by \( N \) hyperplanes?
Almost isometric embeddings of subsets of $\mathbb{S}^d$. 

Picture from Baraniuk, Foucart, Needell, Plan, Wooters
Motivation

- Almost isometric embeddings of subsets of $\mathbb{S}^d$.
- Tessellations with cells small diameter

Every cell of a $\delta$-uniform tessellation of $K$ by hyperplanes has diameter at most $\delta$. If $x$ and $y$ are in the same cell then

$$d(x, y) = |d(x, y) - d_H(x, y)| \leq \delta.$$
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“One-bit” compressed sensing
Tessellations and discrepancy

\[ H_x = \{ z : \langle z, x \rangle > 0 \} \]

\[ W_{xy} = H_x \triangle H_y \]

\[ = \{ z \in S^d : \text{sign}\langle z, x \rangle \neq \text{sign}\langle z, y \rangle \} \]
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\[ \mathbb{P}(\text{sign} \langle z, x \rangle \neq \text{sign} \langle z, y \rangle) \]
\[ = \sigma(W_{xy}) = d(x, y) \]
Tessellations and discrepancy

\[ H_x = \{ z : \langle z, x \rangle > 0 \} \]

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\[ \mathbb{P}(\text{sign} \langle z, x \rangle \neq \text{sign} \langle z, y \rangle) = \sigma(W_{xy}) = d(x, y) \]

\[ \Delta_Z(x, y) = d_H(x, y) - d(x, y) = \frac{\#(Z \cap W_{xy})}{N} - \sigma(W_{xy}) \]

\[ \Delta(Z) = \| \Delta_Z(x, y) \|_\infty = \sup_{x, y \in S^d} \left| \frac{\#(Z \cap W_{xy})}{N} - \sigma(W_{xy}) \right| \]
Lemma (DB, Lacey)

There exists an $N$-point set $Z \subset \mathbb{S}^d$ with

$$\Delta(Z) \leq C_d N^{-\frac{1}{2} - \frac{1}{2d}} \sqrt{\log N}.$$
Lemma (DB, Lacey)

There exists an $N$-point set $Z \subset \mathbb{S}^d$ with

$$\Delta(Z) \leq C_d N^{-\frac{1}{2}} \frac{1}{2^d} \sqrt{\log N}.$$ 

Corollary

This implies that for $\delta > 0$ there exists a $\delta$-uniform tessellation of $\mathbb{S}^d$ by $N$ hyperplanes with

$$N \leq C'_d \delta^{-2+\frac{2}{d+1}} \cdot \left( \log \frac{1}{\delta} \right)^{\frac{d}{d+1}}.$$
Define the $L^2$ discrepancy for wedges

$$\| \Delta Z(x, y) \|_2^2 = \int_{\mathbb{S}^d} \int_{\mathbb{S}^d} \left( \frac{1}{N} \sum_{k=1}^{N} 1_{W_{xy}}(z_k) - \sigma(W_{xy}) \right)^2 \, d\sigma(x) \, d\sigma(y)$$
Define the $L^2$ discrepancy for wedges

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Theorem (Stolarsky principle for the tessellation of the sphere)

For any finite set $Z = \{z_1, \ldots, z_N\} \subset S^d$

$$\| \Delta_Z(x, y) \|^2_2 = \frac{1}{N^2} \sum_{i,j=1}^{N} \left( \frac{1}{2} - d(z_i, z_j) \right)^2 - \int_{S^d} \int_{S^d} \left( \frac{1}{2} - d(x, y) \right)^2 d\sigma(x) d\sigma(y).$$
Frame potential

- $Z = \{z_1, \ldots, z_N\} \subset S^d$ is a frame in $\mathbb{R}^d$ iff there exist $c, C > 0$ such that for any $x \in \mathbb{R}^{d+1}$

\[
c\|x\|^2 \leq \sum_k |\langle x, z_k \rangle|^2 \leq C\|x\|^2.
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- $Z = \{z_1, \ldots, z_N\} \subset \mathbb{S}^d$ is a tight frame iff there exists $A > 0$ such that for any $x \in \mathbb{R}^{d+1}$

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• $Z = \{z_1, \ldots, z_N\} \subset \mathbb{S}^d$ is a tight frame iff there exists $A > 0$ such that for any $x \in \mathbb{R}^{d+1}$

\[ \sum_k |\langle x, z_k \rangle|^2 = A\|x\|^2. \]

**Theorem (Benedetto, Fickus)**

A set $Z = \{z_1, \ldots, z_N\} \subset \mathbb{S}^d$ is a tight frame in $\mathbb{R}^{d+1}$ if and only if $Z$ is a local minimizer of the frame potential:

\[ F(Z) = \sum_{i,j=1}^{N} |\langle z_i, z_j \rangle|^2. \]
\[ Z = \{z_1, \ldots, z_N\} \subset \mathbb{S}^d \text{ is a spherical design of order } t \text{ if it generates a cubature formula, which is exact for all polynomials of degree } t \text{ on } \mathbb{S}^d, \text{ i.e.} \]

\[ \frac{1}{N} \sum_{i=1}^{N} p(z_i) = \int_{\mathbb{S}^d} p(z) d\sigma \text{ whenever } \deg(p) = t. \]
Spherical designs and Korevaar–Meyers conjecture

- $Z = \{z_1, \ldots, z_N\} \subset S^d$ is a spherical design of order $t$ if it generates a cubature formula, which is exact for all polynomials of degree $t$ on $S^d$, i.e.

$$\frac{1}{N} \sum_{i=1}^{N} p(z_i) = \int_{S^d} p(z) d\sigma \quad \text{whenever} \quad \deg(p) = t.$$ 

- Conjecture (Korevaar-Meyers, 1994): There exist spherical designs of order $t$ which consist of $N = \mathcal{O}(t^d)$ points.
Z = \{z_1, \ldots, z_N\} \subset \mathbb{S}^d \text{ is a spherical design of order } t \text{ if it generates a cubature formula, which is exact for all polynomials of degree } t \text{ on } \mathbb{S}^d, \text{ i.e.}

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Conjecture (Korevaar-Meyers, 1994): There exist spherical designs of order } t \text{ which consist of } N = \mathcal{O}(t^d) \text{ points.}

Bondarenko, Radchenko, Viazovska (2012): The conjecture is true! (non-constructive)

Dmitriy Bilyk

Uniform distribution: discrete vs. continuous