Uniform distribution:
approximating continuous objects by discrete ones

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"Introduction to Research" seminar
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Uniform distribution of sequences

- A sequence (x_n) is *uniformly distributed* in $[0, 1]$ iff

  for any interval $I \subset [0,1]$: $\displaystyle \lim_{N\to\infty} \frac{\#\{n \le N : x_n \in I\}}{N} = |I|$

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- For any subsequence (n_k) of integers, the sequence $\{n_k\theta\}$ is uniformly distributed for *a.e.* θ.

Discrepancy

Discrepancy of a sequence

For a sequence $\omega = (\omega_n)_{n=1}^{\infty}$ and an interval $I \subset [0,1]$ consider the quantity

$$\Delta_{N,I} = \sharp\{\omega_n : \omega_n \in I; n \leq N\} - N|I|.$$

Define

$$D_N = \sup_{I \subset [0,1]} \left|\Delta_{N,I}\right|.$$

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A sequence $(\omega_n)_{n=1}^{\infty}$ is u.d. in $[0,1]$ *if and only if*

$$\lim_{N\to\infty} \frac{D_N}{N} = 0.$$

Erdős-Turan inequality

Theorem (Erdős-Turan)

For any sequence $\omega \subset [0,1]$ *we have*

$$D_N(\omega) \lesssim \frac{N}{m} + \sum_{h=1}^{m} \frac{1}{h} \left| \sum_{n=1}^{N} e^{2\pi i h \omega_n} \right|$$

for all natural numbers m.

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- For $\omega = \{n\theta\}$ sharper bounds can be obtained using continued fractions.

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Irregularities of distribution

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Theorem (K. Roth, 1954)

The following are equivalent:
(i) For every $\omega = (\omega_n)_{n=1}^{\infty} \subset [0,1]$,

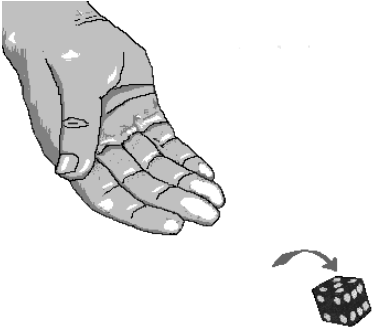
$$D_N(\omega) \gtrsim f(N)$$

for infinitely many values of N.
(ii) For any distribution $\mathcal{P}_N \subset [0,1]^2$ *of* N *points,*

$$\sup_{R-\text{ rectangle}} \Big| \#\mathcal{P}_N \cap R - N \cdot |R| \Big| \gtrsim f(N)$$

Irregularities of Distribution: simplest example

X – roll of a single die



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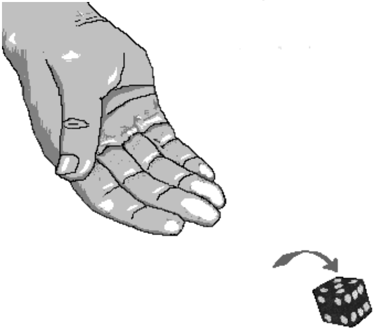
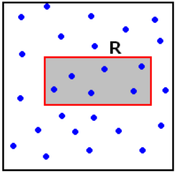


$$\left| X - \mathbb{E}X \right| \geq \frac{1}{2}$$

Geometric Discrepancy

$\mathcal{P}_N$ – a set of N points in $[0,1]^d$
$\mathcal{R}$ –a geometric family (e.g. axis-parallel rectangles, all rectangles, polytopes, balls, convex sets etc.)



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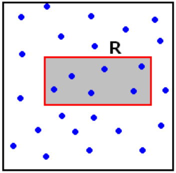

Discrepancy of $\mathcal{P}_N$ with respect to $R \in \mathcal{R}$

$$D(\mathcal{P}_N, R) = \sharp\{\mathcal{P}_N \cap R\} - N \cdot vol\,(R)$$

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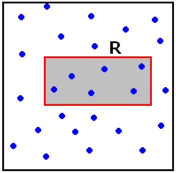

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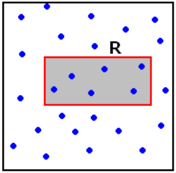

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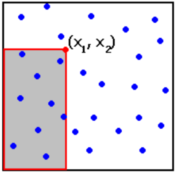
$$D(N) = \inf_{\mathcal{P}_N} D(\mathcal{P}_N)$$

Discrepancy function

Consider a set $\mathcal{P}_N \subset [0,1]^d$ consisting of N points:



Define the discrepancy function of the set $\mathcal{P}_N$ as

$$D_N(x) = \sharp\{\mathcal{P}_N \cap [0,x)\} - N x_1 x_2 \dots x_d$$

Numerical integration

Koksma-Hlawka inequality:

$$\left| \int_{[0,1]^d} f(x)\,dx - \frac{1}{N} \sum_{p \in \mathcal{P}_N} f(p) \right| \lesssim \frac{1}{N} V(f) \cdot \|D_N\|_\infty$$

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- $V(f)$ is the Hardy-Krause variation of f

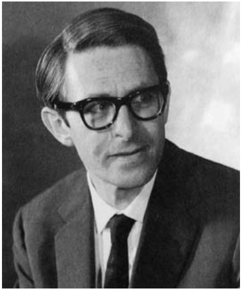
Koksma-Hlawka inequality:

$$\left| \int_{[0,1]^d} f(x)\, dx - \frac{1}{N} \sum_{p \in \mathcal{P}_N} f(p) \right| \lesssim \frac{1}{N} \, \|f_{x_1 \dots x_d}\|_1 \cdot \|D_N\|_\infty$$

- $V(f)$ is the Hardy-Krause variation of f
- $V(f) \;=\; \displaystyle\int_{[0,1]^d} \left| \frac{\partial^d f}{\partial x_1 \partial x_2 \dots \partial x_d} \right| dx_1 \dots dx_d$

  e.g., if $f(x_1, ..., x_d) = \int_{x_1}^1 ... \int_{x_d}^1 \phi(y) dy$

Roth's Theorem



Klaus Roth, October 29, 1925 – November 10, 2015

Theorem (ROTH, K. F. On irregularities of distribution, Mathematika 1 (1954), 73–79.)

There exists $C_d \geq 0$ such that for any N-point set $\mathcal{P}_N \subset [0,1]^d$

$$\|D_N\|_2 \geq C_d (\log N)^{\frac{d-1}{2}}.$$

Roth's Theorem

According to Roth himself, this was his favorite result.

- William Chen (private communication)
- Kenneth Stolarsky (private communication)
- Ben Green (comment on Terry Tao's blog)

12 comments

Comments feed for this article

12 November, 2015 at 9:55 am
Ben Green

I did meet Roth, in Inverness around 7 years ago. I asked him what his favourite proof (among his results was) and he said the lower bound for the L^2 discrepancy of point sets with respect to axis parallel boxes. It is a very elegant argument, nicely described in Bernard Chazelle's book "Discrepancy Theory". Later in his career he became quite interested in the "Heilbronn triangle problem", which came up in conversation the other day: given n points in the unit square, what's the smallest area of triangle they are guaranteed to span. I believe that $n^{-2+o(1)}$ is conjectured, and that Roth was the first to improve on the trivial bound $O(1/n)$. Subsequently bounds of the form $O(n^{-1-c})$ were obtained.

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Roth's Theorem: legacy

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- 4 papers by Roth (On irregularities of distribution. I–IV)
- 10 papers by W.M. Schmidt (On irregularities of distribution. I–X)
- 2 by J. Beck (Note on irregularities of distribution. I–II)
- 4 by W. W. L. Chen (On irregularities of distribution. I–IV)
- 2 by Beck and Chen (Note on irregularities of distribution. I–II)
- a book by Beck and Chen, "Irregularities of distribution".

References

Books

- Kuypers, Niederreiter
  "Uniform distribution of sequences"
- Beck, Chen
  "Irregularities of distribution"
- Drmota, Tichy
  " Sequences, discrepancies and applications"
- Matoušek
  "Geometric discrepancy"
- Dick, Pillichshammer
  "Digital nets and sequences"
- Chazelle
  "Discrepancy method"

Average case: L^p discrepancy, $1 < p < \infty$

Theorem (Roth, 1954 ($p = 2$); Schmidt, 1977 ($1 < p < 2$))

The following estimate holds for all $\mathcal{P}_N \subset [0,1]^d$ with $\#\mathcal{P}_N = N$:

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Theorem (Davenport, 1956 ($d = 2$, $p = 2$); Roth, 1979 ($d \geq 3$, $p = 2$); Frolov, 1980 ($p > 2$, $d = 2$); Chen, 1983 ($p > 2$, $d \geq 3$); Chen, Skriganov, 2000's)

There exist sets $\mathcal{P}_N \subset [0,1]^d$ with

$$\|D_N\|_p \lesssim (\log N)^{\frac{d-1}{2}}$$

L^∞: "worst-case" discrepancy

Conjecture

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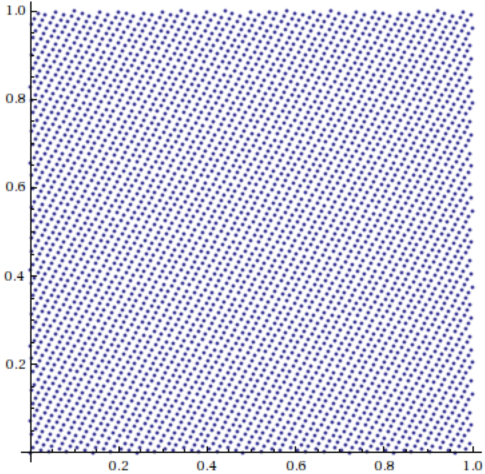
There exist $\mathcal{P}_N \subset [0,1]^2$ with $\|D_N\|_\infty \approx \log N$

Low discrepancy sets

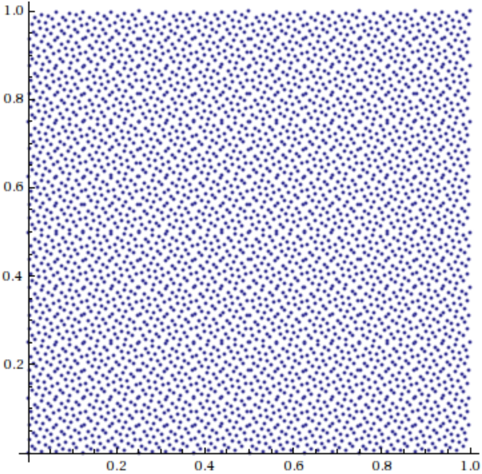


The irrational ($\alpha = \sqrt{2}$) lattice with $N = 2^{12}$ points

$$\big(n/N, \{n\alpha\}\big), \quad n = 0, 1, ..., N-1.$$

Discrepancy $\approx \log N$

Low discrepancy sets

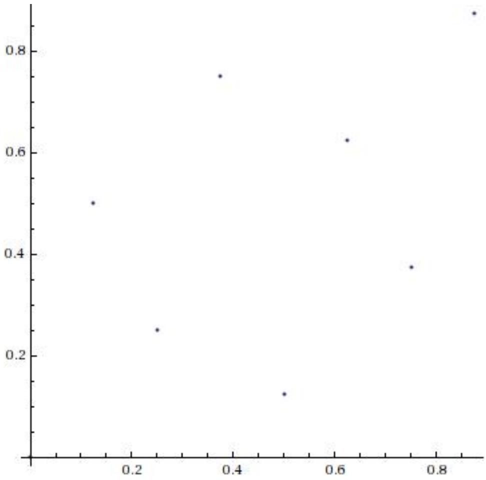


The van der Corput set with $N = 2^n$ points (here $n = 12$)

$$\left(0.x_1 x_2 \ldots x_n,\ 0.x_n x_{n-1} \ldots x_2 x_1\right), \quad x_k = 0 \text{ or } 1.$$

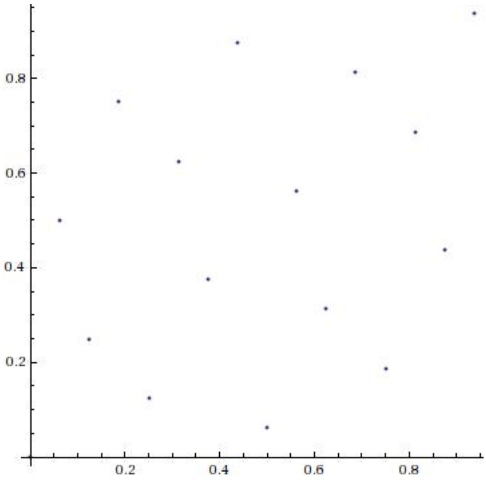
Discrepancy $\approx \log N$

van der Corput set



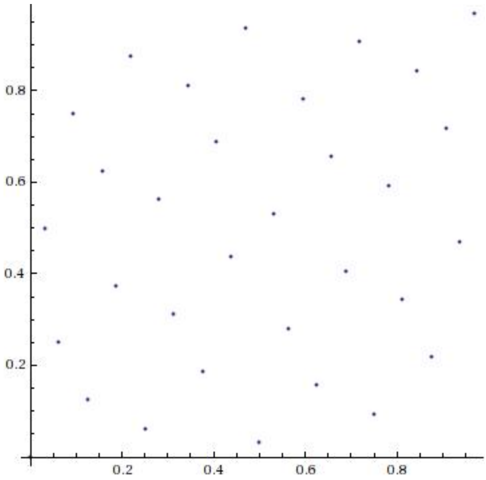
van der Corput set with $N = 2^3$ points

van der Corput set

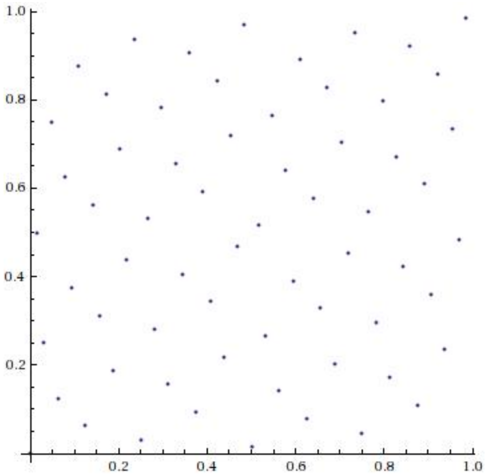


van der Corput set with $N = 2^4$ points

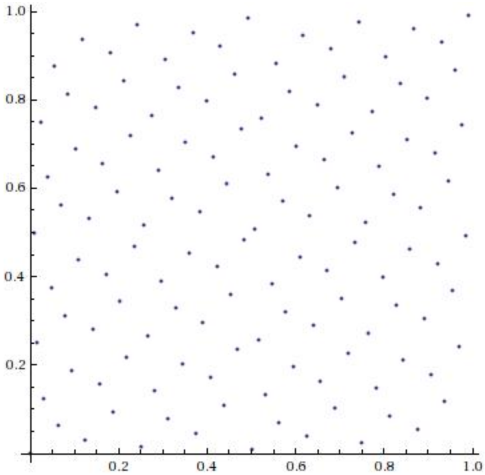
van der Corput set



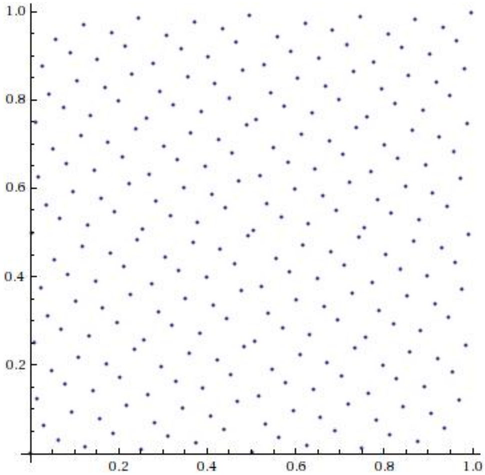
van der Corput set with $N = 2^5$ points



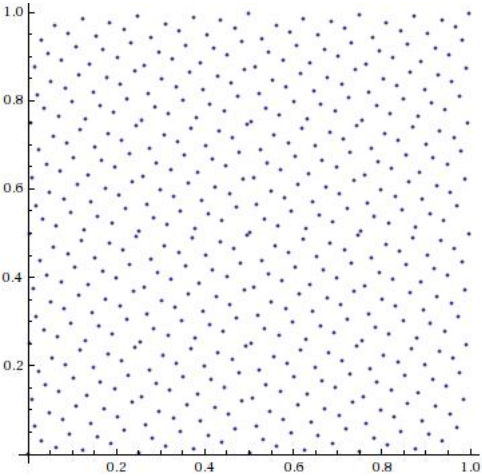
van der Corput set with $N = 2^6$ points



van der Corput set with $N = 2^7$ points

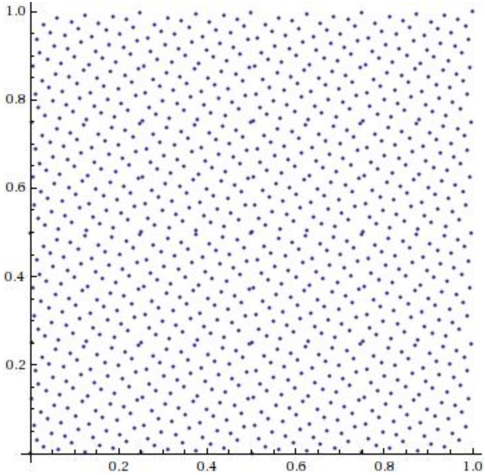


van der Corput set with $N = 2^8$ points



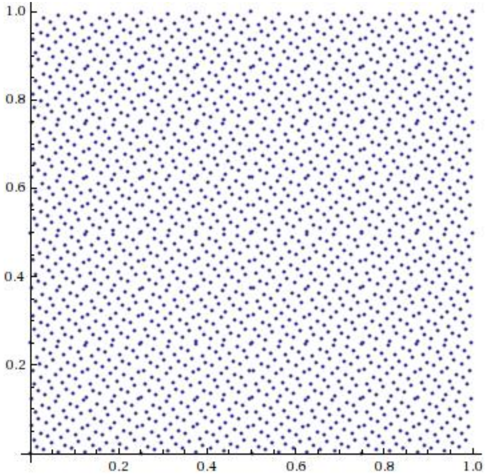
van der Corput set with $N = 2^9$ points

van der Corput set



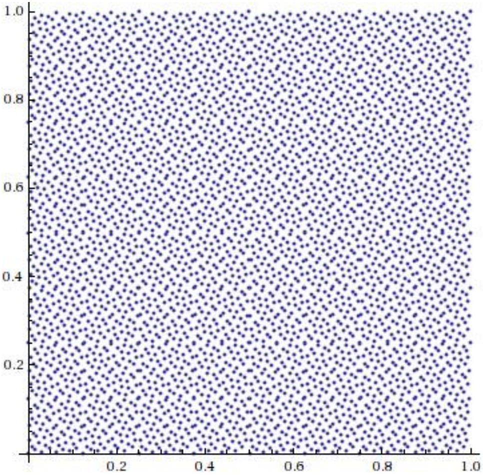
van der Corput set with $N = 2^{10}$ points

van der Corput set



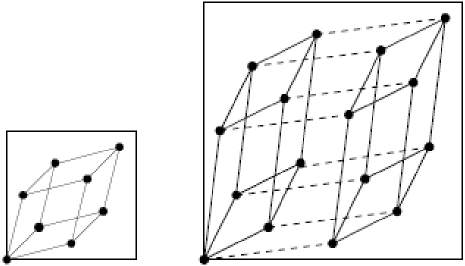
van der Corput set with $N = 2^{11}$ points

van der Corput set



van der Corput set with $N = 2^{12}$ points

van der Corput set



L^∞ estimates

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$\|D_N\|_\infty \gg (\log N)^{\frac{d-1}{2}}$

Theorem (Schmidt, 1972; Halász, 1981)

In dimension $d = 2$ we have $\|D_N\|_\infty \gtrsim \log N$

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There exist $\mathcal{P}_N \subset [0,1]^2$ with $\|D_N\|_\infty \approx \log N$

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$d \geq 3$, Halton, Hammersley (1960):

There exist $\mathcal{P}_N \subset [0,1]^d$ with $\|D_N\|_\infty \lesssim (\log N)^{d-1}$

Conjectures and results

Conjecture 1

$\|D_N\|_\infty \gtrsim (\log N)^{d-1}$

Conjecture 2

$\|D_N\|_\infty \gtrsim (\log N)^{\frac{d}{2}}$

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Conjecture 2

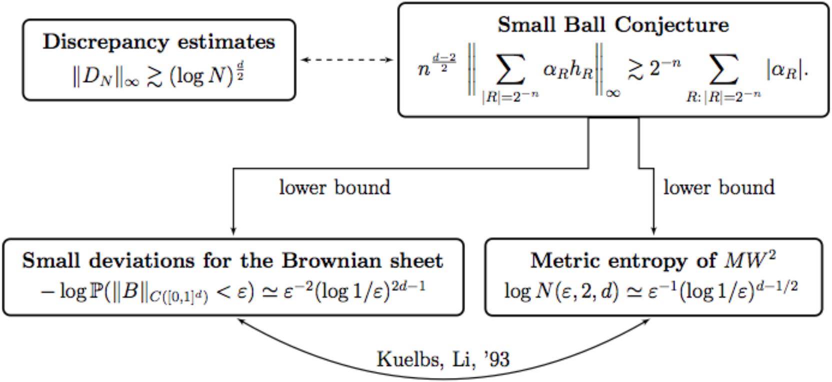
$\|D_N\|_\infty \gtrsim (\log N)^{\frac{d}{2}}$

Theorem (DB, M.Lacey, A.Vagharshakyan, 2008)

For $d \geq 3$ there exists $\eta > 0$ such that the following estimate holds for all N-point distributions $\mathcal{P}_N \subset [0,1]^d$:

$$\|D_N\|_\infty \gtrsim (\log N)^{\frac{d-1}{2}+\eta}.$$

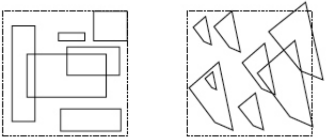
Connections between problems



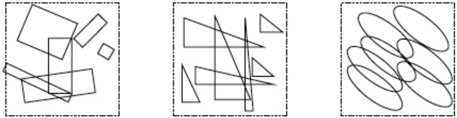
Lower and upper bounds in dimension $d = 2$

LOWER BOUND		UPPER BOUND
	Axis-parallel rectangles	
$D(N, \mathcal{A})$	$\log N$	$\log N$
$D_2(N, \mathcal{A})$	$\sqrt{\log N}$	$\sqrt{\log N}$
	Rotated rectangles	
	$N^{1/4}$	$N^{1/4}\sqrt{\log N}$
	Circles	
	$N^{1/4}$	$N^{1/4}\sqrt{\log N}$
	Convex Sets	
	$N^{1/3}$	$N^{1/3}\log^4 N$

- No rotations: discrepancy $\approx \log N$



- All rotations: discrepancy $\approx N^{1/4}$
  (J. Beck, H. Montgomerry)



- Partial rotations
  (lacunary sets, sets of small Minkowski dimension, etc)
  DB, X.Ma, C. Spencer, J. Pipher (2009-2011)

Higher dimensions: $d \geq 3$

	LOWER BOUND	UPPER BOUND
	Axis-parallel boxes	
L^∞	$(\log N)^{\frac{d-1}{2}+\eta}$	$(\log N)^{d-1}$
L^2	$(\log N)^{\frac{d-1}{2}}$	$(\log N)^{\frac{d-1}{2}}$
	Rotated boxes	
	$N^{\frac{1}{2}-\frac{1}{2d}}$	$N^{\frac{1}{2}-\frac{1}{2d}}\sqrt{\log N}$
	Balls	
	$N^{\frac{1}{2}-\frac{1}{2d}}$	$N^{\frac{1}{2}-\frac{1}{2d}}\sqrt{\log N}$
	Convex Sets	
	$N^{1-\frac{2}{d+1}}$	$N^{1-\frac{2}{d+1}}\log^c N$

Transference: geometric to combinatorial discrepancy

S – a set with N elements, $\mathcal{H}$ – a collection of subset of S,
$\chi : S \to \{-1, 1\}$ – 2-coloring (red-blue)

Combinatorial discrepancy: $$\mathrm{disc}(\mathcal{H}) = \min_{\chi} \max_{A \in \mathcal{H}} \Big| \sum_{x \in A} \chi(x) \Big|$$

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Combinatorial discrepancy generated by geometric systems:

Let $\mathcal{A}$ be a family of measurable sets and $\mathcal{S}_N$ a set of N points.

$$\mathrm{disc}(S_N, \mathcal{A}) = \mathrm{disc}\big(\{S_N \cap A : A \in \mathcal{A}\}\big)$$
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Lemma (Sós; Beck; Lovász, Spencer, Vesztergombi; ...)

Combinatorial discrepancy "is larger than" the geometric discrepancy

$$D(N, \mathcal{A}) \lll \operatorname{disc}(N, \mathcal{A}).$$

Example: Tusnády's problem

Let $\mathcal{R}_d = \{\text{axis-parallel rectangles}\}$.

Tusnády's problem:
What is the asymptotics of $T(N) = \text{disc}(N, \mathcal{R}_d)$ as $N \to \infty$?

- $d = 2$: Matoušek; Beck

$$\log N \lesssim T(N) \lesssim \log^{5/2} N$$

- $d \geq 3$: Nikolov, Matoušek, 2014; Beck

$$(\log N)^{d-1} \lesssim T(N) \lesssim (\log N)^{d+\frac{1}{2}}$$

Spherical cap discrepancy

For $x \in \mathbb{S}^d \subset \mathbb{R}^{d+1}$, $t \in [-1, 1]$ define spherical caps:

$$C(x,t) = \{y \in \mathbb{S}^d : \langle x, y\rangle \geq t\}.$$

For a finite set $Z = \{z_1, z_2, ..., z_N\} \subset \mathbb{S}^d$ define

$$D_{cap}(Z) = \sup_{x \in S^d, t \in [-1,1]} \left| \frac{\#\big(Z \cap C(x,t)\big)}{N} - \sigma\big(C(x,t)\big) \right|.$$

Theorem (Beck)

There exists an N-point set $Z \subset \mathbb{S}^d$ with

$$D_{cap}(Z) \lesssim N^{-\frac{1}{2}-\frac{1}{2d}} \sqrt{\log N}.$$

Theorem (Beck)

For any N-point set $Z \subset \mathbb{S}^d$

$$D_{cap}(Z) \gtrsim N^{-\frac{1}{2}-\frac{1}{2d}}.$$

Spherical caps: L^2

Define the the spherical cap L^2 discrepancy

$$D_{cap}^{(2)} = \left(\int_{\mathbb{S}^{d-1}} \int_{-1}^{1} \left| \frac{\#\big(Z \cap C(x,t)\big)}{N} - \sigma\big(C(x,t)\big) \right|^2 dt\, d\sigma(x) \right)^{\frac{1}{2}}.$$

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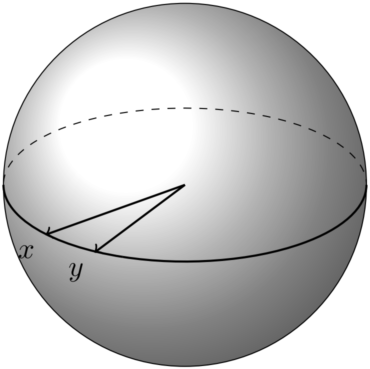
Theorem (Stolarsky invariance principle)

For any finite set $Z = \{z_1, ..., z_N\} \subset \mathbb{S}^{d-1}$

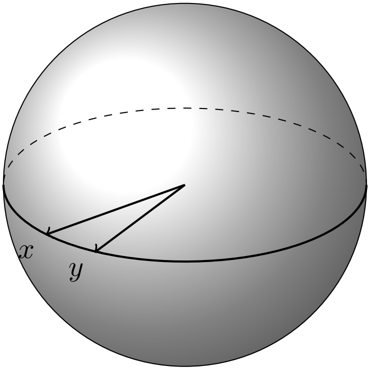
$$\frac{1}{N^2} \sum_{i,j=1}^{N} \|z_i - z_j\| + c_d \left[D_{cap}^{(2)} \right]^2 = \text{const}$$

$$= \int_{\mathbb{S}^{d-1}} \int_{\mathbb{S}^{d-1}} \|x - y\|\, d\sigma(x) d\sigma(y).$$



Let x, $y \in \mathbb{S}^d$ and choose a random hyperplane $z^\perp$, where $z \in \mathbb{S}^d$.

Tessellations of the sphere



Let x, $y \in \mathbb{S}^d$ and choose a random hyperplane $z^{\perp}$, where $z \in \mathbb{S}^d$.

Then

$$\begin{aligned} &\mathbb{P}(z^{\perp} \text{ separates } x \text{ and } y) \\ &= \mathbb{P}(\operatorname{sign}\langle z, x\rangle \neq \operatorname{sign}\langle z, y\rangle) \\ &= d(x, y), \end{aligned}$$

where d is the normalized geodesic distance on the sphere, i.e. $d(x, y) = \frac{\cos^{-1}\langle x,y\rangle}{\pi}$.

Consider a finite set of vectors $Z = \{z_1, z_2, ..., z_N\}$ on the sphere $\mathbb{S}^d$. Define the Hamming distance as

$$d_H(x,y) := \frac{\#\big\{z_k \in Z : \operatorname{sgn}(x \cdot z_k) \neq \operatorname{sgn}(y \cdot z_k)\big\}}{N},$$

i.e. the proportion of hyperplanes $z_k^\perp$ that *separate* x and y.

Uniform tessellations

Define

$$\Delta_Z(x,y) := d_H(x,y) - d(x,y).$$

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We say that Z *is a* δ*-uniform tessellation of* K if

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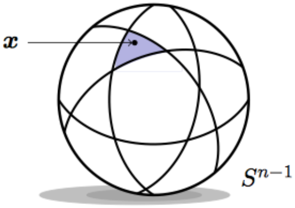
Let $K \subset \mathbb{S}^d$.
We say that Z *is a δ-uniform tessellation of* K if

$$\sup_{x,y\in K} \left|\Delta_Z(x,y)\right| \le \delta.$$

Question:
Given $K \subset \mathbb{S}^d$ and $\delta > 0$, what is the smallest value of N so that there exist a δ-uniform tessellation of K by N hyperplanes?

Motivation



Picture from Baraniuk, Foucart, Needell, Plan, Wooters

- Almost isometric embeddings of subsets of $\mathbb{S}^d$.

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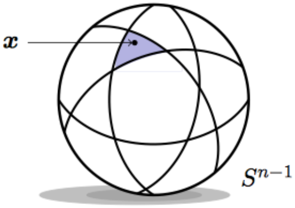

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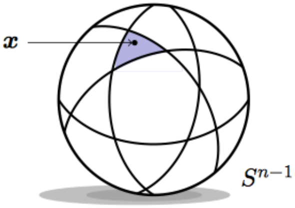
- Almost isometric embeddings of subsets of $\mathbb{S}^d$.
- Tessellations with cells small diameter
  Every cell of a δ-uniform tessellation of K by hyperplanes has diameter at most δ. If x and y are in the same cell then

$$d(x,y) = |d(x,y) - \underbrace{d_H(x,y)}_{=0}| \leq \delta.$$



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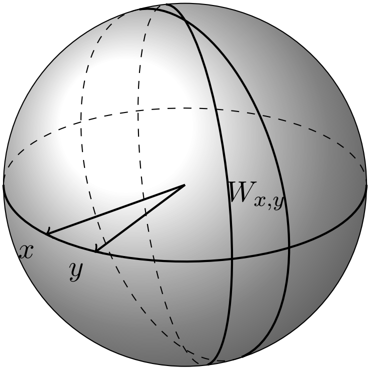
- "One-bit" compressed sensing

Tessellations and discrepancy



$$H_x = \{z \ : \ \langle z, x\rangle > 0\}$$

$$W_{xy} = H_x \triangle H_y$$
$$= \{z \in \mathbb{S}^d : \operatorname{sign}\langle z, x\rangle \neq \operatorname{sign}\langle z, y\rangle\}$$

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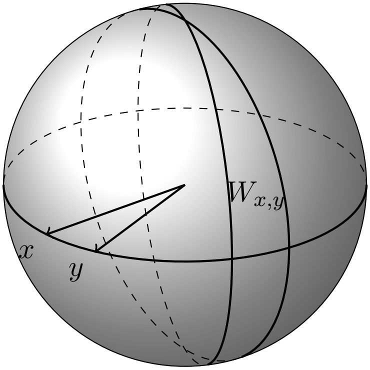

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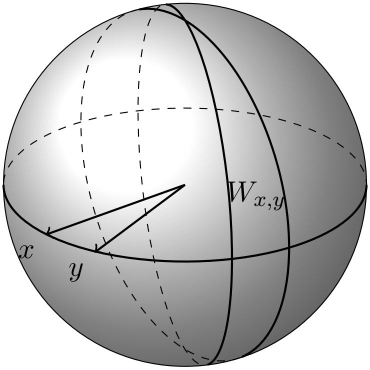

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$$\Delta_Z(x, y) = d_H(x, y) - d(x, y) = \frac{\#(Z \cap W_{xy})}{N} - \sigma(W_{xy})$$

$$\Delta(Z) = \left\|\Delta_Z(x, y)\right\|_\infty = \sup_{x,y\in\mathbb{S}^d} \left| \frac{\#(Z \cap W_{xy})}{N} - \sigma(W_{xy}) \right|.$$

Tessellation/"Wedge" discrepancy

Lemma (DB, Lacey)

There exists an N-point set $Z \subset \mathbb{S}^d$ with

$$\Delta(Z) \le C_d N^{-\frac{1}{2}-\frac{1}{2d}} \sqrt{\log N}.$$

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Corollary

This implies that for $\delta > 0$ there exists a δ-uniform tessellation of $\mathbb{S}^d$ by N hyperplanes with

$$N \leq C_d' \delta^{-2+\frac{2}{d+1}} \cdot \left(\log \frac{1}{\delta}\right)^{\frac{d}{d+1}}.$$

Stolarsky principle for wedge discrepancy

Define the L^2 discrepancy for wedges

$$\left\|\Delta_Z(x,y)\right\|_2^2 = \int_{\mathbb{S}^d}\int_{\mathbb{S}^d}\left(\frac{1}{N}\sum_{k=1}^{N}\mathbf{1}_{W_{xy}}(z_k)-\sigma(W_{xy})\right)^2 d\sigma(x)\, d\sigma(y)$$

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Theorem (Stolarsky principle for the tessellation of the sphere)

For any finite set $Z = \{z_1, \dots, z_N\} \subset \mathbb{S}^d$

$$\left\|\Delta_Z(x,y)\right\|_2^2 =$$

$$\frac{1}{N^2}\sum_{i,j=1}^{N} \left(\frac{1}{2} - d(z_i, z_j)\right)^2 - \int\limits_{\mathbb{S}^d}\int\limits_{\mathbb{S}^d} \left(\frac{1}{2} - d(x,y)\right)^2 d\sigma(x)\, d\sigma(y).$$

Frame potential

- $Z = \{z_1, \dots, z_N\} \subset \mathbb{S}^d$ is a frame in $\mathbb{R}^d$ iff there exist $c, C > 0$ such that for any $x \in \mathbb{R}^{d+1}$

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Theorem (Benedetto, Fickus)

A set $Z = \{z_1, \dots, z_N\} \subset \mathbb{S}^d$ is a tight frame in $\mathbb{R}^{d+1}$ if and only if Z is a local minimizer of the frame potential:

$$F(Z) = \sum_{i,j=1}^{N} |\langle z_i, z_j \rangle|^2.$$

Spherical designs and Korevaar–Meyers conjecture

- $Z = \{z_1, \ldots, z_N\} \subset \mathbb{S}^d$ is a spherical design of order t if it generates a cubature formula, which is exact for all polynomials of degree t on $\mathbb{S}^d$, i.e.

$$\frac{1}{N}\sum_{i=1}^{N} p(z_i) = \int_{\mathbb{S}^d} p(z)\,d\sigma \quad \text{whenever} \quad \deg(p) = t.$$

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- Conjecture (Korevaar-Meyers, 1994): There exist spherical designs of order t which consist of $N = \mathcal{O}(t^d)$ points.

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- Bondarenko, Radchenko, Viazovska (2012): The conjecture is true! (non-constructive)