Dmitriy Bilyk
Georgia Institute of Technology

Atlanta, GA
April 4, 2007
Let $\mathcal{P}_N$ be a subset of $[0, 1]^d$ of cardinality $N$ and let $R \subset [0, 1]^d$ be a rectangle with sides parallel to the axis.
Let $\mathcal{P}_N$ be a subset of $[0, 1]^d$ of cardinality $N$ and let $R \subset [0, 1]^d$ be a rectangle with sides parallel to the axis.

$$D(\mathcal{P}_N, R) = \#\{\mathcal{P}_N \cap R\} - N \cdot \text{vol}(R)$$
Let $\mathcal{P}_N$ be a subset of $[0, 1]^d$ of cardinality $N$ and let $R \subset [0, 1]^d$ be a rectangle with sides parallel to the axis.

\[
D(\mathcal{P}_N, R) = \#\{\mathcal{P}_N \cap R\} - N \cdot \text{vol}(R)
\]

\[
D(N) = \inf_{\mathcal{P}_N} \sup_R |D(\mathcal{P}_N, R)|
\]

Can $D(N)$ be bounded? NO (van Aardenne-Ehrenfest; Roth)
Let $\mathcal{P}_N$ be a subset of $[0, 1]^d$ of cardinality $N$ and let $R \subset [0, 1]^d$ be a rectangle with sides parallel to the axis.

$$D(\mathcal{P}_N, R) = \#\{\mathcal{P}_N \cap R\} - N \cdot \text{vol}(R)$$

$$D(N) = \inf_{\mathcal{P}_N} \sup_{R} |D(\mathcal{P}_N, R)|$$

Can $D(N)$ be bounded?
Let $\mathcal{P}_N$ be a subset of $[0, 1]^d$ of cardinality $N$ and let $R \subset [0, 1]^d$ be a rectangle with sides parallel to the axis.

$$D(\mathcal{P}_N, R) = \# \{ \mathcal{P}_N \cap R \} - N \cdot \text{vol}(R)$$

$$D(N) = \inf_{\mathcal{P}_N} \sup_{R} |D(\mathcal{P}_N, R)|$$

Can $D(N)$ be bounded?

NO (van Aardenne-Ehrenfest; Roth)
Enough to consider rectangles with a vertex at the origin

$$\text{Discrepancy function}$$

$$D_N(x) = \sum_{i=1}^{d} P_{N \cap [0,x)}(x_i) - Nx_1x_2 \ldots x_d$$

Lower estimates for $$\|D_N\|_p$$
Discrepancy function

Enough to consider rectangles with a vertex at the origin

$$D_N(x) = \#\{P_N \cap [0, x)\} - Nx_1x_2 \ldots x_d$$
Discrepancy function

Enough to consider rectangles with a vertex at the origin

\[ D_N(x) = \#\{ \mathcal{P}_N \cap [0, x) \} - Nx_1x_2\ldots x_d \]

Lower estimates for \( \|D_N\|_p \)
An example

Consider a $\sqrt{N} \times \sqrt{N}$ lattice

Thus $\|D_N\|_\infty \gtrsim N^{1/2}$
Consider a $\sqrt{N} \times \sqrt{N}$ lattice

Thus $\|D_N\|_\infty \gg N^{1/2}$
Consider a $\sqrt{N} \times \sqrt{N}$ lattice

$$|D_N(P) - D_N(Q)| \approx N \cdot \frac{1}{\sqrt{N}} = \sqrt{N}$$
Consider a $\sqrt{N} \times \sqrt{N}$ lattice

$|D_N(P) - D_N(Q)| \approx N \cdot \frac{1}{\sqrt{N}} = \sqrt{N}$

Thus $\|D_N\|_\infty \gtrsim N^{\frac{1}{2}}$
$L^p$ estimates, $1 < p < \infty$

**Theorem**

$$\|D_N\|_p \gtrsim (\log N)^{\frac{d-1}{2}}$$

*Roth ($p=2$), Schmidt*
$L^p$ estimates, $1 < p < \infty$

**Theorem**

$$\|D_N\|_p \geq (\log N)^{\frac{d-1}{2}}$$

*Roth* ($p=2$), *Schmidt*

**Theorem**

There exist $\mathcal{P}_N \subset [0, 1]^d$ with

$$\|D_N\|_p \approx (\log N)^{\frac{d-1}{2}}$$

*(Davenport, Roth, Frolov, Chen)*
$L^\infty$ estimates

Conjecture

$\|D_N\|_\infty \gg (\log N)^\frac{d-1}{2}$
Conjecture

\[ \|D_N\|_\infty \gg (\log N)^{\frac{d-1}{2}} \]

Theorem

(Schmidt) For \( d = 2 \) we have \( \|D_N\|_\infty \gtrsim \log N \)
Conjecture

\[ \| D_N \|_\infty \gg (\log N)^{\frac{d-1}{2}} \]

Theorem

(Schmidt) For \( d = 2 \) we have

\[ \| D_N \|_\infty \gtrsim \log N \]

\( d = 2 \)

van der Corput: There exist \( \mathcal{P}_N \subset [0, 1]^2 \) with

\[ \| D_N \|_\infty \approx \log N \]
\[ \|D_N\|_\infty \gg (\log N)^{\frac{d-1}{2}} \]

**Conjecture**

**Theorem**

*(Schmidt)* For \( d = 2 \) we have \( \|D_N\|_\infty \gtrsim \log N \)

\( d = 2 \)

van der Corput: There exist \( \mathcal{P}_N \subset [0, 1]^2 \) with \( \|D_N\|_\infty \approx \log N \)

\( d \geq 3 \)

Halasza: There exist \( \mathcal{P}_N \subset [0, 1]^d \) with \( \|D_N\|_\infty \approx (\log N)^{d-1} \)
Conjecture 1

\[ \| D_N \|_\infty \gtrsim (\log N)^{d-1} \]
Conjectures

Conjecture 1
\[ \| D_N \|_\infty \gtrsim (\log N)^{d-1} \]

Conjecture 2
\[ \| D_N \|_\infty \gtrsim (\log N)^\frac{d}{2} \]
Consider a sequence \( \{a_n\}_{n=1}^{\infty} \subset [0, 1] \) and the quantity

\[
\Delta_N(x) = \#\{a_n : a_n \leq x; n \leq N\} - Nx.
\]

Can \( \sup_N \|\Delta_N\|_\infty \) be bounded?
The ‘dynamical’ problem

Consider a sequence \( \{a_n\}_{n=1}^{\infty} \subset [0, 1] \) and the quantity

\[
\Delta_N(x) = \#\{a_n : a_n \leq x; n \leq N\} - Nx.
\]

Can \( \sup_N \|\Delta_N\|_\infty \) be bounded?

Theorem (K. Roth)

The following are equivalent:

(i) For every sequence \( \{a_n\}_{n=1}^{\infty} \subset [0, 1] \), we have the estimate

\[
\|\Delta_N\|_\infty \gtrsim f(N) \text{ for infinitely many } N.
\]

(ii) For any \( N \) point distribution \( P_N \subset [0, 1]^2 \), \( \|D_N\|_\infty \gtrsim f(N) \).
Consider a sequence \( \{a_n\}_{n=1}^{\infty} \subset [0, 1] \) and the quantity

\[
\Delta_N(x) = \# \{a_n : a_n \leq x; n \leq N \} - Nx.
\]

Can \( \sup_N \| \Delta_N \|_\infty \) be bounded?

**Theorem (K. Roth)**

The following are equivalent:

(i) For every sequence \( \{a_n\}_{n=1}^{\infty} \subset [0, 1] \), we have the estimate

\[
\| \Delta_N \|_\infty \gtrsim f(N) \text{ for infinitely many } N.
\]

(ii) For any \( N \) point distribution \( \mathcal{P}_N \subset [0, 1]^2 \), \( \| D_N \|_\infty \gtrsim f(N) \).

\[
\| D_N \|_\infty \approx \max_{k=1,\ldots,N} \| \Delta_k \|_\infty.
\]
The ‘dynamical’ problem

Consider a sequence \( \{a_n\}_{n=1}^\infty \subset [0, 1] \) and the quantity

\[
\Delta_N(x) = \#\{a_n : a_n \leq x; n \leq N\} - Nx.
\]

Can sup\( N \|\Delta_N\|_\infty \) be bounded?

Theorem (K. Roth)

The following are equivalent:

(i) For every sequence \( \{a_n\}_{n=1}^\infty \subset [0, 1] \), we have the estimate

\[
\|\Delta_N\|_\infty \gtrsim f(N)
\]

for infinitely many \( N \).

(ii) For any \( N \) point distribution \( \mathcal{P}_N \subset [0, 1]^2 \),

\[
\|D_N\|_\infty \gtrsim f(N).
\]

\[
\|D_N\|_\infty \approx \max_{k=1,\ldots,N} \|\Delta_k\|_\infty.
\]

\[
\{a_n\}_{n=1}^\infty \rightarrow \left\{ \left( \frac{k}{N}, a_k \right) \right\}_{k=1}^N
\]
Two basic types of behavior of the discrepancy function

discrepancy about \( \log n \)
Two basic types of behavior of the discrepancy function

- Discrepancy about $\log n$
- Discrepancy about $n^{1/4}$
Dyadic intervals are intervals of the form \([k2^q, (k + 1)2^q]\).
Dyadic intervals are intervals of the form \([k2^q, (k + 1)2^q]\).

For a dyadic interval \(I\): \(h_I = -1_{\text{left}} + 1_{\text{right}}\), Haar functions with \(L^\infty\) normalization.
Dyadic intervals are intervals of the form $[k2^q, (k + 1)2^q]$.

For a dyadic interval $I$: $h_I = -1_{left} + 1_{right}$, Haar functions with $L^\infty$ normalization.

In higher dimensions: for a rectangle $R = I_1 \times \cdots \times I_d$

$$h_R(x_1, \ldots, x_d) := h_{I_1}(x_1) \cdot \cdots \cdot h_{I_d}(x_d)$$
All rectangles are in the unit cube in $\mathbb{R}^d$. 

\[ H_d^n = \{ (r_1, r_2, \ldots, r_d) \in \mathbb{N}^d : r_1 + \cdots + r_d = n \} \]
All rectangles are in the unit cube in $\mathbb{R}^d$.

Choose $n$ such that $2N \leq 2^n \leq 4N$ and consider dyadic rectangles of volume $2^{-n}$. 

\[ H_d^n = \{ (r_1, r_2, \ldots, r_d) \in \mathbb{N}^d : r_1 + \cdots + r_d = n \} \]
All rectangles are in the unit cube in $\mathbb{R}^d$.

Choose $n$ such that $2N \leq 2^n \leq 4N$ and consider dyadic rectangles of volume $2^{-n}$.

$$\mathcal{H}_n^d = \{(r_1, r_2, \ldots, r_d) \in \mathbb{N}^d : r_1 + \cdots + r_d = n\}$$
All rectangles are in the unit cube in $\mathbb{R}^d$. Choose $n$ such that $2N \leq 2^n \leq 4N$ and consider dyadic rectangles of volume $2^{-n}$.

$$\mathbb{H}_n^d = \{(r_1, r_2, \ldots, r_d) \in \mathbb{N}^d : r_1 + \cdots + r_d = n\}$$

**Definition**

For $\vec{r} \in \mathbb{H}_n^d$, call $f$ an $\vec{r}$ function iff it is of the form

$$f = \sum_{R : |R_j|=2^{-r_j}} \varepsilon_R h_R, \quad \varepsilon_R \in \{\pm 1\}.$$
Call a rectangle $R$ good if $R \cap \mathcal{P}_n = \emptyset$.
Call a rectangle $R$ good if $R \cap \mathcal{P}_n = \emptyset$.

At least half of the rectangles are good.
Call a rectangle $R$ good if $R \cap \mathcal{P}_n = \emptyset$.

At least half of the rectangles are good.

If $R$ is good, then

$$\langle D_N, h_R \rangle = N \langle x_1 x_2 \ldots x_d, h_R \rangle = N \cdot \frac{1}{4} |R|^2 \approx 2^{-n}.$$
Roth’s Orthogonal Function Method: Proof

- Call a rectangle $R$ good if $R \cap \mathcal{P}_n = \emptyset$
- At least half of the rectangles are good.
- If $R$ is good, then

$$\langle D_N, h_R \rangle = N \langle x_1 x_2 \ldots x_d, h_R \rangle = N \cdot \frac{1}{4} |R|^2 \approx 2^{-n}$$

- For each $\vec{r}$, construct an $\vec{r}$ function of the form

$$f_r = \sum_{R \text{ is good}} h_R + \sum_{R \text{ is bad}} \text{sign} \langle D_N, h_R \rangle \ h_R$$
Call a rectangle $R$ good if $R \cap \mathcal{P}_n = \emptyset$.

At least half of the rectangles are good.

If $R$ is good, then

$$\langle D_N, h_R \rangle = N\langle x_1 x_2 \ldots x_d, h_R \rangle = N \cdot \frac{1}{4} |R|^2 \approx 2^{-n}$$

For each $\vec{r}$, construct an $\vec{r}$ function of the form

$$f_r = \sum_{R \text{ good}} h_R + \sum_{R \text{ bad}} \text{sign} \langle D_N, h_R \rangle h_R$$

Then $\langle D_N, f_r \rangle \gtrsim 1$. 
Construct the test function $F = \sum_{\vec{r} \in \mathbb{H}_n} f_r$. 
Construct the test function \( F = \sum_{\vec{r} \in \mathbb{H}_n^d} f_r. \)
\[
\#\{\vec{r} \in \mathbb{H}_n^d\} \approx n^{d-1} \approx (\log N)^{d-1}.
\]
Roth’s Orthogonal Function Method: Proof

- Construct the test function $F = \sum_{\vec{r} \in \mathbb{H}^d} f_r$.
- $\#\{\vec{r} \in \mathbb{H}^d\} \approx n^{d-1} \approx (\log N)^{d-1}$.
- $\|F\|_2 \approx (\log N)^{\frac{d-1}{2}}$. 

---

Dmitriy Bilyk
The Discrepancy Theory: Classical and New Results
• Construct the test function $F = \sum_{\vec{r} \in \mathbb{H}_n^d} f_r$.
• $\#\{\vec{r} \in \mathbb{H}_n^d\} \approx n^{d-1} \approx (\log N)^{d-1}$.
• $\|F\|_2 \approx (\log N)^{\frac{d-1}{2}}$.
• $\langle D_N, F \rangle \gtrsim (\log N)^{d-1}$.

Thus

$$\|D_N\|_2 \geq \langle D_N, f_r \rangle \|F\|_2 \gtrsim (\log N)^{\frac{d-1}{2}}.$$
Roth’s Orthogonal Function Method: Proof

- Construct the test function \( F = \sum_{\vec{r} \in \mathbb{H}^d} f_r \).
- \( \#\{\vec{r} \in \mathbb{H}^d\} \approx n^{d-1} \approx (\log N)^{d-1} \).
- \( \|F\|_2 \approx (\log N)^{\frac{d-1}{2}} \).
- \( \langle D_N, F \rangle \gtrsim (\log N)^{d-1} \).
- Thus

\[
\|D_N\|_2 \geq \frac{\langle D_N, f_r \rangle}{\|F\|_2} \gtrsim (\log N)^{\frac{d-1}{2}}
\]
"Bit reversal sequence"

Construct the sequence $r(i)$ as follows: if $i = a_0 + a_1 2^1 + a_2 2^2 + ..., \text{ then}$

$$r(i) = \frac{a_0}{2} + \frac{a_1}{2^2} + \frac{a_2}{2^3} + \ldots$$

For example, for $i = 13 = 1101_2$, we have
$$r(13) = 0.1011_2.$$
Van der Corput set

"Bit reversal sequence"

Construct the sequence $r(i)$ as follows: if

$$i = a_0 + a_1 2^1 + a_2 2^2 + ...,$$

then

$$r(i) = \frac{a_0}{2} + \frac{a_1}{2^2} + \frac{a_2}{2^3} + ...$$

e.g., for $i = 13 = 1101_2$, we have $r(13) = 0.1011_2$. 
Van der Corput set

"Bit reversal sequence"

Construct the sequence $r(i)$ as follows: if $i = a_0 + a_1 2^1 + a_2 2^2 + \ldots$, then

$$r(i) = \frac{a_0}{2} + \frac{a_1}{2^2} + \frac{a_2}{2^3} + \ldots$$

E.g., for $i = 13 = 1101_2$, we have $r(13) = 0.1011_2$. Then define the point set $\mathcal{P}_N \subset [0, 1]^2$ as $\left\{ \left( \frac{i}{N}, r(i) \right) \right\}_{i=0}^{N-1}$
Van der Corput set

"Bit reversal sequence"

Construct the sequence $r(i)$ as follows: if $i = a_0 + a_1 2^1 + a_2 2^2 + \ldots$, then

$$r(i) = \frac{a_0}{2} + \frac{a_1}{2^2} + \frac{a_2}{2^3} + \ldots$$

e.g., for $i = 13 = 1101_2$, we have $r(13) = 0.1011_2$. Then define the point set $\mathcal{P}_N \subset [0, 1]^2$ as $\{(\frac{i}{N}, r(i))\}_{i=0}^{N-1}$.

![Diagram](image)

Fig. 2.2. The Van der Corput set for $n = 8$ (a) and for $n = 64$ (b).
Theorem

The discrepancy of the $N$-element Van der Corput set satisfies

$$\|D_N\|_\infty \approx \log N.$$  

Claim 1.

For $R = [0, a) \times I$, where $I$ is dyadic, we have $|D(P, R)| \leq 1$. 

Dmitriy Bilyk
The Discrepancy Theory: Classical and New Results
Van der Corput set has low discrepancy

**Theorem**

*The discrepancy of the $N$-element Van der Corput set satisfies*

$$\|D_N\|_\infty \approx \log N.$$  

**Claim 1.**

For $R = [0, a) \times I$, where $I$ is dyadic, we have $|D(\mathcal{P}, R)| \leq 1$.  

![Diagram showing the region $R$ and $I$.]
Claim 1.

For $R = [0, a) \times I$, where $I$ is dyadic, we have $|D(\mathcal{P}, R)| \leq 1$. 
Van der Corput set has low discrepancy

Claim 1.
For $R = [0, a) \times I$, where $I$ is dyadic, we have $|D(\mathcal{P}, R)| \leq 1$. 

- Consider the points of $\mathcal{P}$ which live in $R$.
- For these points $r(i) \in I$.
- This means that the first $q$ binary digits of $r(i)$ are fixed.
- Thus the last $q$ digits of $i$ are fixed ($i = \text{const} \mod 2^q$).
- So, the $x$-coordinates are equally spaced with step $2^q/N$.
- Each rectangle of area $1/N$ has exactly 1 point and thus discrepancy zero.
Claim 1.
For $R = [0, a) \times I$, where $I$ is dyadic, we have $|D(\mathcal{P}, R)| \leq 1$.

Consider the points of $\mathcal{P}$ which live in $R$
Claim 1.

For \( R = [0, a) \times I \), where \( I \) is dyadic, we have \( |D(\mathcal{P}, R)| \leq 1 \).

Consider the points of \( \mathcal{P} \) which live in \( R \).

For these points \( r(i) \in I \).
Claim 1.

For $R = [0, a) \times I$, where $I$ is dyadic, we have $|D(\mathcal{P}, R)| \leq 1$.

- Consider the points of $\mathcal{P}$ which live in $R$
- For these points $r(i) \in I$
- This means that the first $q$ binary digits of $r(i)$ are fixed
Claim 1.
For $R = [0, a) \times I$, where $I$ is dyadic, we have $|D(\mathcal{P}, R)| \leq 1$.

- Consider the points of $\mathcal{P}$ which live in $R$
- For these points $r(i) \in I$
- This means that the first $q$ binary digits of $r(i)$ are fixed
- Thus the last $q$ digits of $i$ are fixed ($i = const \mod 2^q$)
Van der Corput set has low discrepancy

Claim 1.

For $R = [0, a) \times I$, where $I$ is dyadic, we have $|D(\mathcal{P}, R)| \leq 1$.

- Consider the points of $\mathcal{P}$ which live in $R$
- For these points $r(i) \in I$
- This means that the first $q$ binary digits of $r(i)$ are fixed
- Thus the last $q$ digits of $i$ are fixed ($i = const \mod 2^q$)
- So, the $x$-coordinates are equally spaced with step $2^q/N$. 
Claim 1.

For $R = [0, a) \times I$, where $I$ is dyadic, we have $|D(\mathcal{P}, R)| \leq 1$.

- Consider the points of $\mathcal{P}$ which live in $R$
- For these points $r(i) \in I$
- This means that the first $q$ binary digits of $r(i)$ are fixed
- Thus the last $q$ digits of $i$ are fixed ($i = \text{const mod } 2^q$)
- So, the $x$-coordinates are equally spaced with step $2^q/N$.
- Each rectangle of area $1/N$ has exactly 1 point and thus discrepancy zero.
Claim 2.

Any $R = [0, x) \times [0, y)$ can be decomposed into $\leq \lceil \log_2 N \rceil$ rectangles as in Claim 1 plus a rectangle $M$ with $|D(P, M)| \leq 1$.
Van der Corput set has low discrepancy

Claim 2.

Any \( R = [0, x) \times [0, y) \) can be decomposed into \( \leq \lceil \log_2 N \rceil \) rectangles as in Claim 1 plus a rectangle \( M \) with \( |D(P, M)| \leq 1 \).

\[ M \]

\[ y_0 \]

\[ (x, y) \]

The area of \( M \) is at most \( y - y_0 \leq 2^{-n} \leq 1/N \).

\( M \) contains at most 1 point of \( P \).

Dmitriy Bilyk

The Discrepancy Theory: Classical and New Results
Claim 2.

Any $R = [0, x) \times [0, y)$ can be decomposed into $\leq \lceil \log_2 N \rceil$ rectangles as in Claim 1 plus a rectangle $M$ with $|D(P, M)| \leq 1$.

- Let $n$ be the smallest integer with $2^n \geq N$. 
Van der Corput set has low discrepancy

Claim 2.

Any \( R = [0, x) \times [0, y) \) can be decomposed into \( \leq \lceil \log_2 N \rceil \) rectangles as in Claim 1 plus a rectangle \( M \) with \( |D(\mathcal{P}, M)| \leq 1 \).

Let \( n \) be the smallest integer with \( 2^n \geq N \).

\( y_0 \) is the largest multiple of \( 2^{-n} \) not exceeding \( y \).
Claim 2.

Any \( R = [0, x) \times [0, y) \) can be decomposed into \( \leq \lceil \log_2 N \rceil \) rectangles as in Claim 1 plus a rectangle \( M \) with \( |D(P, M)| \leq 1 \).

- Let \( n \) be the smallest integer with \( 2^n \geq N \).
- \( y_0 \) is the largest multiple of \( 2^{-n} \) not exceeding \( y \).
- \( M = [0, x) \times [y_0, y) \).
Claim 2.

Any $R = [0, x) \times [0, y)$ can be decomposed into $\leq \lceil \log_2 N \rceil$ rectangles as in Claim 1 plus a rectangle $M$ with $|D(P, M)| \leq 1$.

- Let $n$ be the smallest integer with $2^n \geq N$.
- $y_0$ is the largest multiple of $2^{-n}$ not exceeding $y$.
- $M = [0, x) \times [y_0, y)$.
- The area of $M$ is at most $y - y_0 \leq 2^{-n} \leq 1/N$. 
Claim 2.

Any \( R = [0, x) \times [0, y) \) can be decomposed into \( \leq \lceil \log_2 N \rceil \) rectangles as in Claim 1 plus a rectangle \( M \) with \( |D(\mathcal{P}, M)| \leq 1 \).

- Let \( n \) be the smallest integer with \( 2^n \geq N \).
- \( y_0 \) is the largest multiple of \( 2^{-n} \) not exceeding \( y \).
- \( M = [0, x) \times [y_0, y) \).
- The area of \( M \) is at most \( y - y_0 \leq 2^{-n} \leq 1/N \).
- \( M \) contains at most 1 point of \( \mathcal{P} \).
Halton-Hammersely set

Let $p_1, p_2, \ldots, p_{d-1}$ be distinct primes.
Halton-Hammersely set

Let \( p_1, p_2, \ldots, p_{d-1} \) be distinct primes.
Then define the point set \( \mathcal{P}_N \subset [0, 1]^d \) as
\[
\left\{ \left( \frac{i}{N}, r_{p_1}(i), r_{p_2}(i), \ldots, r_{d-1}(i) \right) \right\}_{i=0}^{N-1}
\]
Generalization to higher dimensions

**Halton-Hammersley set**

Let $p_1, p_2, \ldots, p_{d-1}$ be distinct primes.
Then define the point set $\mathcal{P}_N \subset [0, 1]^d$ as
\[
\left\{ \left( \frac{i}{N}, r_{p_1}(i), r_{p_2}(i), \ldots, r_{d-1}(i) \right) \right\}_{i=0}^{N-1}
\]

**Theorem**

*The discrepancy function of the Halton-Hammersley set satisfies*

\[
\|D_N\|_\infty \approx \log^{d-1} N.
\]
Another low discrepancy sequence

Example

Let $\alpha$ be an irrational number and let $\{x\}$ denote the fractional part of $x$. 

Theorem

If the partial quotients of the continued fraction of $\alpha$ are bounded, then the discrepancy function of this set satisfies $\|D_N\|_{\infty} \approx \log N$.

In particular works for quadratic irrationalities $\alpha = u + \sqrt{v}$.

The idea goes as far back as 1904 (Lerch)
Another low discrepancy sequence

Example

Let $\alpha$ be an irrational number and let $\{x\}$ denote the fractional part of $x$. Define $\mathcal{P}_N = \left\{ \left( \frac{i}{N}, \{i\alpha\} \right) \right\}_{i=0}^{N-1}$

Theorem

If the partial quotients of the continued fraction of $\alpha$ are bounded, then the discrepancy function of this set satisfies $\|D_N\|_{\infty} \approx \log N$.

In particular works for quadratic irrationalities $\alpha = u + \sqrt{v}$.

The idea goes as far back as 1904 (Lerch).
Another low discrepancy sequence

Example

Let $\alpha$ be an irrational number and let $\{x\}$ denote the fractional part of $x$. Define $\mathcal{P}_N = \left\{ \left( \frac{i}{N}, \{i\alpha\} \right) \right\}_{i=0}^{N-1}$

Theorem

*If the partial quotients of the continued fraction of $\alpha$ are bounded, then the discrepancy function of this set satisfies* $\|D_N\|_\infty \approx \log N$. 

In particular, works for quadratic irrationalities $\alpha = u + \sqrt{v}$. 

The idea goes as far back as 1904 (Lerch).
Another low discrepancy sequence

Example

Let \( \alpha \) be an irrational number and let \( \{x\} \) denote the fractional part of \( x \). Define \( \mathcal{P}_N = \{ \left( \frac{i}{N}, \{i\alpha\} \right) \}_{i=0}^{N-1} \)

Theorem

If the partial quotients of the continued fraction of \( \alpha \) are bounded, then the discrepancy function of this set satisfies \( \|D_N\|_\infty \approx \log N \).

- In particular works for quadratic irrationalities \( \alpha = u + \sqrt{v} \).
Another low discrepancy sequence

**Example**

Let \( \alpha \) be an irrational number and let \( \{ x \} \) denote the fractional part of \( x \). Define \( \mathcal{P}_N = \left\{ \left( \frac{i}{N}, \{ i\alpha \} \right) \right\}_{i=0}^{N-1} \)

**Theorem**

*If the partial quotients of the continued fraction of \( \alpha \) are bounded, then the discrepancy function of this set satisfies \( \| D_N \|_\infty \approx \log N \).*

- In particular works for quadratic irrationalities \( \alpha = u + \sqrt{v} \).
- The idea goes as far back as 1904 (Lerch)
We are interested in non-trivial lower bounds for the 'hyperbolic' sums of Haar functions:

\[ \left\| \sum_{|R|=2^{-n}} \alpha_R h_R \right\|_\infty \]
The ‘Trivial Bound’

$L^2$ estimate

\[ n^{\frac{d-1}{2}} \left\| \sum_{|R|=2^{-n}} \alpha_R h_R \right\|_{\infty} \geq 2^{-n} \sum_{|R|=2^{-n}} |\alpha_R| \]
The ‘Trivial Bound’

$L^2$ estimate

\[ n^{\frac{d-1}{2}} \left\| \sum_{|R|=2^{-n}} \alpha_R h_R \right\|_\infty \gtrsim 2^{-n} \sum_{|R|=2^{-n}} |\alpha_R| \]

\[ 2^{-n} \sum_{|R|=2^{-n}} |\alpha_R| = \left\| \sum_{|R|=2^{-n}} |\alpha_R| h_R \right\|_1 \]
The ‘Trivial Bound’

$L^2$ estimate

$$n^{\frac{d-1}{2}} \left\| \sum_{|R|=2^{-n}} \alpha_R h_R \right\|_\infty \gtrsim 2^{-n} \sum_{|R|=2^{-n}} |\alpha_R|$$

$$2^{-n} \sum_{|R|=2^{-n}} |\alpha_R| = \left\| \sum_{|R|=2^{-n}} |\alpha_R| |h_R| \right\|_1$$

$$\leq n^{(d-1)/2} \left\| \left[ \sum_{|R|=2^{-n}} |\alpha_R|^2 h_R^2 \right]^{1/2} \right\|_1$$

↑ About $n^{d-1}$ rectangles can overlap
The ‘Trivial Bound’

$L^2$ estimate

\[ n^{\frac{d-1}{2}} \left\| \sum_{|R|=2^{-n}} \alpha_R \ h_R \right\|_\infty \gtrsim 2^{-n} \sum_{|R|=2^{-n}} |\alpha_R| \]

\[ 2^{-n} \sum_{|R|=2^{-n}} |\alpha_R| = \left\| \sum_{|R|=2^{-n}} |\alpha_R||h_R| \right\|_1 \]

\[ \leq n^{(d-1)/2} \left\| \left[ \sum_{|R|=2^{-n}} |\alpha_R|^2 \ h_R^2 \right]^{1/2} \right\|_2 \]
The ‘Trivial Bound’

$L^2$ estimate

\[ n^{\frac{d-1}{2}} \left\| \sum_{|R|=2^{-n}} \alpha_R h_R \right\|_{\infty} \gtrsim 2^{-n} \sum_{|R|=2^{-n}} |\alpha_R| \]

\[ 2^{-n} \sum_{|R|=2^{-n}} |\alpha_R| = \left\| \sum_{|R|=2^{-n}} |\alpha_R| h_R \right\|_1 \]

\[ \leq n^{(d-1)/2} \left\| \left[ \sum_{|R|=2^{-n}} |\alpha_R|^2 h_R^2 \right]^{1/2} \right\|_2 \]

\[ = n^{(d-1)/2} \left( \sum_{|R|=2^{-n}} |\alpha_R|^2 2^{-n} \right)^{1/2} \]
The ‘Trivial Bound’

$L^2$ estimate

\[ n^{\frac{d-1}{2}} \left\| \sum_{|R|=2^{-n}} \alpha_R \, h_R \right\|_\infty \gtrsim 2^{-n} \sum_{|R|=2^{-n}} |\alpha_R| \]

\[ 2^{-n} \sum_{|R|=2^{-n}} |\alpha_R| = \left\| \sum_{|R|=2^{-n}} |\alpha_R| \, h_R \right\|_1 \]

\[ \leq n^{(d-1)/2} \left\| \left[ \sum_{|R|=2^{-n}} |\alpha_R|^2 \, h_R^2 \right]^{1/2} \right\|_2 \]

\[ = n^{(d-1)/2} \left\| \sum_{|R|=2^{-n}} \alpha_R \, h_R \right\|_2 \]
The ‘Trivial Bound’

\[ L^2 \text{ estimate} \]

\[ n^{\frac{d-1}{2}} \left\| \sum_{|R|=2^{-n}} \alpha_R h_R \right\|_\infty \gtrsim 2^{-n} \sum_{|R|=2^{-n}} |\alpha_R| \]

\[ 2^{-n} \sum_{|R|=2^{-n}} |\alpha_R| = \left\| \sum_{|R|=2^{-n}} |\alpha_R| h_R \right\|_1 \]

\[ \leq n^{(d-1)/2} \left\| \left[ \sum_{|R|=2^{-n}} |\alpha_R|^2 h_R^2 \right]^{1/2} \right\|_2 \]

\[ \leq n^{(d-1)/2} \left\| \sum_{|R|=2^{-n}} \alpha_R h_R \right\|_\infty \]
Small Ball Conjecture

For dimensions $d \geq 2$, we have

$$2^{-n} \sum_{|R|=2^{-n}} |\alpha_R| \lesssim n^{\frac{1}{2}}(d-2) \left\| \sum_{|R|=2^{-n}} \alpha_R h_R \right\|_\infty$$
Small Ball Conjecture

For dimensions $d \geq 2$, we have

$$2^{-n} \sum_{|R|=2^{-n}} |\alpha_R| \lesssim n^{1/2}(d-2) \left\| \sum_{|R|=2^{-n}} \alpha_R h_R \right\|_\infty$$

This conjecture is related to:

Irregularities of Distribution
Approximation Theory
Probability Theory
The Conjecture

Small Ball Conjecture

For dimensions $d \geq 2$, we have

$$2^{-n} \sum_{|R|=2^{-n}} |\alpha_R| \lesssim n^{\frac{1}{2}(d-2)} \left\| \sum_{|R|=2^{-n}} \alpha_R h_R \right\|_{\infty}$$

This conjecture is related to:

- Irregularities of Distribution
The Conjecture

Small Ball Conjecture

For dimensions $d \geq 2$, we have

$$2^{-n} \sum_{|R|=2^{-n}} |\alpha_R| \lesssim n^{\frac{1}{2}}(d-2) \left\| \sum_{|R|=2^{-n}} \alpha_R h_R \right\|_{\infty}$$

This conjecture is related to:

- Irregularities of Distribution
- Approximation Theory
The Conjecture

Small Ball Conjecture

For dimensions $d \geq 2$, we have

$$2^{-n} \sum_{|R|=2^{-n}} |\alpha_R| \lesssim n^{\frac{1}{2}}(d-2) \left\| \sum_{|R|=2^{-n}} \alpha_R h_R \right\|_{\infty}$$

This conjecture is related to:

- Irregularities of Distribution
- Approximation Theory
- Probability Theory
The Small Ball Conjecture and Discrepancy

**Small Ball Conjecture**

For dimensions $d \geq 2$, we have

$$2^{-n} \sum_{|R|=2^{-n}} |\alpha_R| \lesssim n^{\frac{1}{2}(d-2)} \left\| \sum_{|R|=2^{-n}} \alpha_R h_R \right\|_{\infty}$$

Notice that, in both conjectures, one gains a square root over the $L^2$ estimate.
The Small Ball Conjecture and Discrepancy

**Small Ball Conjecture**

For dimensions $d \geq 2$, we have

$$2^{-n} \sum_{|R|=2^{-n}} |\alpha_R| \lesssim \left( n \right)^{\frac{1}{2} (d-2)} \left\| \sum_{|R|=2^{-n}} \alpha_R h_R \right\|_{\infty}$$

**Conjecture 2**

$$\left\| D_N \right\|_{\infty} \gtrsim (\log N)^{\frac{d}{2}}$$
The Small Ball Conjecture and Discrepancy

**Small Ball Conjecture**

For dimensions $d \geq 2$, we have

$$2^{-n} \sum_{|R|=2^{-n}} |\alpha_R| \lesssim n^{\frac{1}{2}(d-2)} \left\| \sum_{|R|=2^{-n}} \alpha_R h_R \right\|_{\infty}$$

**Conjecture 2**

$$\|D_N\|_{\infty} \gtrsim (\log N)^{\frac{d}{2}}$$

- Notice that, in both conjectures, one gains a square root over the $L^2$ estimate.
The Conjecture

Small Ball Conjecture

For dimensions \( d \geq 2 \), we have

\[
2^{-n} \sum_{|R|=2^{-n}} |\alpha_R| \lesssim n^{\frac{1}{2}} (d-2) \left\| \sum_{|R|=2^{-n}} \alpha_R h_R \right\|_{\infty}
\]

Random choice of sign shows this is sharp.

Talagrand has proved this in \( d=2 \). Another proof by Temlyakov.
The Conjecture

Small Ball Conjecture

For dimensions $d \geq 2$, we have

$$2^{-n} \sum_{|R|=2^{-n}} |\alpha_R| \lesssim n^{\frac{1}{2}} (d-2) \left\| \sum_{|R|=2^{-n}} \alpha_R h_R \right\|_\infty$$

- Random choice of sign shows this is sharp.
Small Ball Conjecture

For dimensions $d \geq 2$, we have

$$2^{-n} \sum_{|R|=2^{-n}} |\alpha_R| \lesssim n^{\frac{1}{2}} (d-2) \left\| \sum_{|R|=2^{-n}} \alpha_R h_R \right\|_\infty$$

- Random choice of sign shows this is sharp.
- Talagrand has proved this in $d = 2$. Another proof by Temlyakov.
Main result

Theorem (DB & Michael Lacey)

In dimension $d = 3$ there is a $\eta > 0$ for which we have

$$2^{-n} \sum_{|R|=2^{-n}} |\alpha_R| \lesssim n^{1-\eta} \left\| \sum_{|R|=2^{-n}} \alpha_R h_R \right\|_{\infty}$$

(1)

Previously known: József Beck (1989):

$$n^{-\eta} \leftarrow (\log n)^{-1/8}.$$
Proof of Talagrand’s Theorem, following Temlyakov and Schmidt

\[ 2^{-n} \sum_{|R|=2^{-n}} |\alpha_R| \lesssim \left\| \sum_{|R|=2^{-n}} \alpha_R h_R \right\|_{\infty} \]

The method of proof is by duality. Use Riesz products.

In Dimension 2, there is a 'Product Rule': If \(|R|=|R'|\), then,

\[ h_R \cdot h_{R'} \text{ is either } \]

\[ 0, 1, R \pm h_R \cap R'. \]
Proof of Talagrand’s Theorem, following Temlyakov and Schmidt

Theorem

In dimension 2, we have

\[ 2^{-n} \sum_{|R|=2^{-n}} |\alpha_R| \lesssim \left\| \sum_{|R|=2^{-n}} \alpha_R h_R \right\|_\infty \]

- The method of proof is by duality.
Proof of Talagrand’s Theorem, following Temlyakov and Schmidt

Theorem

In dimension 2, we have

\[
2^{-n} \sum_{|R|=2^{-n}} |\alpha_R| \lesssim \left\| \sum_{|R|=2^{-n}} \alpha_R h_R \right\|_{\infty}
\]

- The method of proof is by duality.
- Use Riesz products.
The method of proof is by duality.

- Use Riesz products.

In Dimension 2, there is a ‘Product Rule’: If $|R| = |R'|$, then, $h_R \cdot h_{R'}$ is either

$$0, \quad 1_R, \quad \pm h_{R \cap R'}.$$
Product Rule Fails in higher dimensions!!!

Dmitriy Bilyk

The Discrepancy Theory: Classical and New Results
Proof of Talagrand’s Theorem, following Temlyakov and Schmidt

\[ H \stackrel{\text{def}}{=} \sum_{|R|=2^{-n}} \alpha_R h_R \]
Proof of Talagrand’s Theorem, following Temlyakov and Schmidt

\[ H \overset{\text{def}}{=} \sum_{|R|=2^{-n}} \alpha_R h_R \]

\[ f_k \overset{\text{def}}{=} \sum_{|R_1|=2^{-k}, |R_2|=2^{-n+k}} \text{sgn}(\alpha_R) h_R, \quad 0 \leq k \leq n, \]
$H \overset{\text{def}}{=} \sum_{|R|=2^{-n}} \alpha_R h_R$

$f_k \overset{\text{def}}{=} \sum_{|R_1|=2^{-k}, |R_2|=2^{-n+k}} \text{sgn}(\alpha_R) h_R, \quad 0 \leq k \leq n$

$F \overset{\text{def}}{=} \prod_{k=0}^{n} \left(1 + \frac{1}{2} f_k\right)$
Proof of Talagrand’s Theorem, following Temlyakov and Schmidt

\[ H \overset{\text{def}}{=} \sum_{|R|=2^{-n}} \alpha_R h_R \]

\[ f_k \overset{\text{def}}{=} \sum_{|R_1|=2^{-k}, |R_2|=2^{-n+k}} \text{sgn}(\alpha_R) h_R, \quad 0 \leq k \leq n, \]

\[ F \overset{\text{def}}{=} \prod_{k=0}^{n} \left(1 + \frac{1}{2} f_k \right) \]

\[ F \geq 0 \quad \& \quad \int F = 1 \quad \Rightarrow \quad \|F\|_1 = 1 \]
Proof of Talagrand’s Theorem, following Temlyakov and Schmidt

\[ H \overset{\text{def}}{=} \sum_{|R|=2^{-n}} \alpha_R h_R \]

\[ f_k \overset{\text{def}}{=} \sum_{|R_1|=2^{-k}, |R_2|=2^{-n+k}} \text{sgn}(\alpha_R) h_R, \quad 0 \leq k \leq n, \]

\[ F \overset{\text{def}}{=} \prod_{k=0}^{n} \left(1 + \frac{1}{2} f_k\right) \]

\[ F \geq 0 \quad \& \quad \int F = 1 \quad \Rightarrow \quad \|F\|_1 = 1 \]

\[ \|H\|_\infty \geq \langle H, F \rangle = \sum_{k=0}^{n} \langle H, f_k \rangle = 2^{-n-1} \sum_{|R|=2^{-n}} |\alpha_R| \]
Theorem (DB & M. Lacey)

There is a choice of $0 < \eta < \frac{1}{2}$ for which the following estimate holds for all collections $\mathcal{P}_N \subset [0, 1]^3$:

$$\|D_N\|_{\infty} \gtrsim (\log N)^{1+\eta}.$$
A New Result for the Discrepancy Function in three dimensions

**Theorem (DB & M. Lacey)**

There is a choice of $0 < \eta < \frac{1}{2}$ for which the following estimate holds for all collections $\mathcal{P}_N \subset [0, 1]^3$:

$$\|D_N\|_\infty \gtrsim (\log N)^{1+\eta}.$$ 

Previously known: József Beck (1989):

$$(\log N)^{\eta} \leftarrow (\log \log n)^{1/8}.$$
A New Result for the Discrepancy Function in higher dimensions

Theorem (DB, M. Lacey, A. Vagharshakyan)

For $d \geq 3$ there is a choice of $0 < \eta(d) < \frac{1}{2}$ for which the following estimate holds for all collections $\mathcal{P}_N \subset [0, 1]^d$:

$$\|D_N\|_{\infty} \gtrsim (\log N)^{\frac{d-1}{2}} + \eta(d).$$
A New Result for the Discrepancy Function in higher dimensions

Theorem (DB, M.Lacey, A.Vagharshakyan)

For $d \geq 3$ there is a choice of $0 < \eta(d) < \frac{1}{2}$ for which the following estimate holds for all collections $\mathcal{P}_N \subset [0,1]^d$:

$$\|D_N\|_\infty \gtrsim (\log N)^{\frac{d-1}{2}} + \eta(d).$$

$$\eta(d) \approx \frac{1}{d}$$
Theorem (DB, Michael Lacey & Armen Vagharshakyan)

In dimensions $d \geq 3$ there is a $\eta(d) > 0$, such that for all choices of coefficients $\alpha_R \in \{\pm 1\}$ we have

$$\left\| \sum_{|R|=2^{-n}} \alpha_R h_R \right\|_{\infty} \gtrsim n^{\frac{d-1}{2}} + \eta(d)$$

(2)

For $d = 3$, $\eta = \frac{1}{10} - \varepsilon$. 
Theorem (DB, Michael Lacey & Armen Vagharshakyan)

In dimensions $d \geq 3$ there is a $\eta(d) > 0$, such that for all choices of coefficients $\alpha_R \in \{\pm 1\}$ we have

$$\left\| \sum_{|R|=2^{-n}} \alpha_R h_R \right\|_\infty \gtrsim n^{\frac{d-1}{2}} + \eta(d)$$

(2)

- For $d = 3$, $\eta = \frac{1}{10} - \varepsilon$.
- In higher dimensions, $\eta(d) \approx \frac{1}{d}$. 
Unrestricted inequality in all dimensions

Theorem (DB, Michael Lacey & Armen Vagharshakyan)

In dimensions $d \geq 3$ there is a $\eta(d) > 0$, such that for all choices of coefficients $\alpha_R$ we have

$$n^{\frac{d-1}{2}} - \eta(d) \left\| \sum_{|R|=2^{-n}} \alpha_R h_R \right\|_{\infty} \gtrsim 2^{-n} \sum_{|R|=2^{-n}} |\alpha_R|$$

(3)
Let $MD = D_{x_1} \cdots D_{x_d}$ be the mixed derivative operator.
Let $MD = D_{x_1} \cdots D_{x_d}$ be the mixed derivative operator.

Define $\text{Ball}_s \overset{\text{def}}{=} \{ f : \|f\|_s + \|MD f\|_s \leq 1 \}$
Let \( MD = D_{x_1} \cdots D_{x_d} \) be the mixed derivative operator.

Define \( \text{Ball}_s \) as \( \{ f : \| f \|_s + \| MD f \|_s \leq 1 \} \)

Define \( N_s(\epsilon) := \) least number of \( L^\infty \) balls of radius \( \epsilon \) needed to cover \( \text{Ball}_s \)
Let $MD = D_{x_1} \cdots D_{x_d}$ be the mixed derivative operator.

Define $Ball_s \overset{\text{def}}{=} \{ f : \|f\|_s + \|MD f\|_s \leq 1 \}$

Define $N_s(\epsilon) := \text{least number of } L^\infty \text{ balls of radius } \epsilon \text{ needed to cover } Ball_s$

**Conjecture**

*For $d \geq 2$, one has the estimate* $\log N_2(\epsilon) \approx \frac{1}{\epsilon} \left( \log \frac{1}{\epsilon} \right)^{d-1/2}$
Covering Numbers of Mixed Derivative Spaces

- Let $MD = D_{x_1} \cdots D_{x_d}$ be the mixed derivative operator.
- Define $\text{Ball}_s \overset{\text{def}}{=} \{ f : \|f\|_s + \|MD f\|_s \leq 1 \}$
- Define $N_s(\epsilon) := \text{least number of } L^\infty \text{ balls of radius } \epsilon \text{ needed to cover } \text{Ball}_s$

**Conjecture**

*For $d \geq 2$, one has the estimate* $\log N_2(\epsilon) \approx \frac{1}{\epsilon} \left( \log \frac{1}{\epsilon} \right)^{d-1/2}$

- Upper bound is known
Let \( MD = D_{x_1} \cdots D_{x_d} \) be the mixed derivative operator.

Define \( \text{Ball}_s \overset{\text{def}}{=} \{ f : \| f \|_s + \| MD f \|_s \leq 1 \} \)

Define \( N_s(\epsilon) := \) least number of \( L^\infty \) balls of radius \( \epsilon \) needed to cover \( \text{Ball}_s \)

**Conjecture**

For \( d \geq 2 \), one has the estimate
\[
\log N_2(\epsilon) \approx \frac{1}{\epsilon} \left( \log \frac{1}{\epsilon} \right)^{d-1/2}
\]

- Upper bound is known
- Small Ball inequality for smooth wavelets implies the lower bound (Talagrand)
Let \( MD = D_{x_1} \cdots D_{x_d} \) be the mixed derivative operator.

Define \( \text{Ball}_s \overset{\text{def}}{=} \{ f : \| f \|_s + \| MD f \|_s \leq 1 \} \)

Define \( N_s(\epsilon) := \) least number of \( L^\infty \) balls of radius \( \epsilon \) needed to cover \( \text{Ball}_s \)

**Conjecture**

For \( d \geq 2 \), one has the estimate

\[
\log N_2(\epsilon) \approx \frac{1}{\epsilon} \left( \log \frac{1}{\epsilon} \right)^{d-1/2}
\]

Upper bound is known

Small Ball inequality for smooth wavelets implies the lower bound (Talagrand)

Small Ball Inequality for Haars implies a bound for \( N_1(\epsilon) \)
Let $B : [0, 1]^d \rightarrow \mathbb{R}$ be the Brownian Sheet.
Let $B : [0,1]^d \rightarrow \mathbb{R}$ be the Brownian Sheet.

**Theorem (Kuelbs, Li)**

$- \log \mathbb{P}(\|B\|_{C[0,1]^d} < \epsilon) \approx \epsilon^{-2} \left( \log \frac{1}{\epsilon} \right)^\beta$ \quad \text{iff} \quad \log N_2(\epsilon) \approx \epsilon^{-1} \left( \log \frac{1}{\epsilon} \right)^{\beta/2}$
Let $B : [0, 1]^d \rightarrow \mathbb{R}$ be the Brownian Sheet.

**Theorem (Kuelbs, Li)**

$$- \log \mathbb{P}(\|B\|_{C[0,1]^d} < \epsilon) \approx \epsilon^{-2} \left( \log \frac{1}{\epsilon} \right)^{\beta} \quad \text{iff} \quad \log N_2(\epsilon) \approx \epsilon^{-1} \left( \log \frac{1}{\epsilon} \right)^{\beta/2}$$

**Small Ball Problem**

For $d \geq 2$, we have

$$- \log \mathbb{P}(\|B\|_{C[0,1]^d} < \epsilon) \approx \epsilon^{-2} \left( \log \frac{1}{\epsilon} \right)^{2d-1}$$
A Good Introduction to the subject

Geometric Discrepancy
An Illustrated Guide

Jiří Matoušek
For fixed $x$, $\mathbb{E} \left| \sum_{R \in \mathcal{R}_n} \pm h_R(x) \right| \approx n^{(d-1)/2}$.
Sharpness of the Conjecture

- For fixed $x$, $\mathbb{E} \left| \sum_{R \in \mathcal{R}_n} \pm h_R(x) \right| \sim n^{(d-1)/2}$.
- By well known inequality for supremum of sub-Gaussian rvs,

$$\mathbb{E} \sup_x \left| \sum_{R \in \mathcal{R}_n} \pm h_R(x) \right| \lesssim \sqrt{\log 2 n} \sup_x \mathbb{E} \left| \sum_{R \in \mathcal{R}_n} \pm h_R \right|$$

$$\lesssim n^{d/2}$$
For fixed $x$, $\mathbb{E}\left|\sum_{R \in \mathcal{R}_n} \pm h_R(x)\right| \simeq n^{(d-1)/2}$.

By well known inequality for supremum of sub–Gaussian rvs,

$$\mathbb{E} \sup_x \left|\sum_{R \in \mathcal{R}_n} \pm h_R(x)\right| \lesssim \sqrt{\log 2^n} \sup_x \mathbb{E} \left|\sum_{R \in \mathcal{R}_n} \pm h_R\right| \lesssim n^{d/2}$$

On the other side,

$$2^{-n} \sum_{|R| = 2^{-n}} |\pm 1| \simeq n^{d-1}$$