1. Suppose the population of raccoons in the city is 100 & the population of raccoons in the nearby forest is 300. Suppose we also know that 10% of the raccoons move from the woods to the city, & 5% of the raccoons move from the city to the woods per year.

@ Set up a transition matrix to describe this phenomenon.

Recall: We should think about each entry a_{ij} in the transition matrix as the probability that the system will move from state i to state j. (p. 284)

Let \( W(K) \) denote the pop of raccoons in the woods & \( C(K) \) denote the pop of raccoons in the city.

Our transition matrix takes us from time \( K \) to time \( K + 1 \):

\[
\begin{bmatrix}
W(K + 1) \\
C(K + 1)
\end{bmatrix} = \begin{bmatrix} W & C \\
C(K) & P_{11} & P_{12} \\
C(K) & P_{21} & P_{22}
\end{bmatrix} \begin{bmatrix} W(K) \\
C(K)
\end{bmatrix}
\]

So, \( P_{11} \): probability that the raccoon will stay in the woods when it is the woods & \( P_{11} = 0.9 \).

\( P_{12} \): prob. that rac. will move from city to woods \( P_{12} = 0.05 \).

\( P_{21} \): prob. that rac. will move from woods to city \( P_{21} = 0.1 \).

\( P_{22} \): prob. that raccoons in city will stay in city \( P_{22} = 0.95 \). So, \( \begin{bmatrix} 0.9 & 0.05 \\
0.1 & 0.95
\end{bmatrix} \) is our transition matrix.
(b) Is $T$ a regular stochastic matrix?

Recall: A square matrix $A$ is called a stochastic matrix if each of its columns is a probability vector (i.e., the entries of each column add up to 1). (pg. 285).

A stochastic matrix $A$ is called regular if $A$ or some positive power of $A$ has all positive entries.

$$T = \begin{bmatrix} 0.9 & 0.05 \\ 0.1 & 0.95 \end{bmatrix}$$ is stochastic

Since $0.9 + 0.1 = 1$ and $0.05 + 0.95 = 1$,

$$\therefore T \text{ is regular since all of } T\text{'s entries are positive.}$$

(c) Does $T$ have a steady-state vector? If so, what is it?

Recall: If $P$ is a regular transition matrix for a Markov chain, then exists a probability vector $q$, s.t. $Pq = q$ (i.e., $q$ is an eigenvector corresponding to $\lambda = 1$ and $q$'s entries sum to 1). This vector $q$ is called the steady-state vector. (pg. 288).

So, since $T$ is regular, we know $T$ does have a steady-state vector. Let's find it:
\[ \lambda = 1: \begin{bmatrix} 0.9 & 0.05 \\ 0.1 & 0.95 \end{bmatrix} \]

\[ \begin{bmatrix} 0.1 & 0.05 \\ 0.1 & -0.05 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \]

\[ \begin{align*}
\frac{1}{6}x &= \frac{1}{20}y \quad \Rightarrow \quad x = \frac{1}{2}t \\
y &= t
\end{align*} \]

So, \( \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} \) solves this system.

But, we need that our steady-state vector is an eigenvector corresponding to \( \lambda = 1 \) AND its entries have to sum to 1.

\[ \frac{1}{2}t + t = 1 \quad \Rightarrow \quad \frac{3}{2}t = 1 \quad \Rightarrow \quad t = \frac{2}{3} \]

So, \( \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} \) \( \frac{2}{3} \) solves this system and its a probability vector. \( \therefore \begin{bmatrix} \frac{1}{2} \\ \frac{2}{3} \end{bmatrix} \) is the steady-state vector.

\[ \text{0 In the long term, how will the population of raccoons in the city and woods be distributed?} \]

Recall: If \( q \) is a steady-state vector for a regular Markov chain, then for any initial probability vector \( x_0 \), \( \lim_{k \to \infty} p^k x_0 = q \), where \( p \) is the transition matrix for this chain. (Theorem 4.12, p. 283).
Since our steady-state vector is \[ \begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \end{bmatrix} \], we know that in the long term, \( \frac{1}{3} \) of raccoons will be in the woods, and \( \frac{2}{3} \) of them will be in the city.

Recall: We know \( x_n = P^n x_0 \).

Here, \( P = \begin{bmatrix} 0.9 & 0.05 \\ 0.1 & 0.45 \end{bmatrix} \) and \( x_0 = \begin{bmatrix} 34 \\ 34 \end{bmatrix} \).

So, we need to find \( A^{20} \). To do this, we should diagonalize \( A \). (Since \( A^{20} = P D^{20} P^{-1} \)).

\[
\begin{vmatrix} 0.9 - \lambda & 0.05 \\ 0.1 & 0.45 - \lambda \end{vmatrix} = (0.9 - \lambda)(0.45 - \lambda) - \frac{1}{10} \cdot \frac{1}{20} = \lambda^2 - \frac{3}{50} \lambda + \frac{19}{200} = 0
\]

I know \( \lambda = 1 \) is an eigenvalue b/c reg. stochastic. So:

\[
\lambda - 1 - \frac{\sqrt{\lambda^2 - \frac{3}{20} \lambda + \frac{19}{200}}}{\lambda - 20} - \frac{\sqrt{\lambda^2 - \frac{3}{20} \lambda + \frac{19}{200}}}{\lambda - 20} - \frac{1}{10} \lambda + \frac{19}{200} = 0
\]

So, \( \lambda = 1 \) and \( \lambda = \frac{17}{20} \) are my eigenvalues.

By \( \odot \) I know \( \begin{bmatrix} 13 \\ 12 \end{bmatrix} \) is an eigenvector corresponding to \( \lambda = 1 \).
\[
\begin{bmatrix}
0.9 & -0.85 & 0.05 & 0 \\
0.1 & 0.95 & -0.85 & 0 \\
0.05 & 0.05 & 0 & 0 \\
0.1 & 0.1 & 0 & 0 \\
\end{bmatrix}
\]

\[
\frac{1}{10} x = -\frac{1}{10} y, \quad x = -t, \quad y = t.
\]

So \([-1]^T\) solves this system. \([-1]^T\) is an eigenvector corresponding to \(\lambda = \frac{1}{10}\).

So, \(P = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}\), \(D = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{10} \end{bmatrix}\).

\(P^{-1} = \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix} \frac{1}{3} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ -\frac{2}{3} & \frac{1}{3} \end{bmatrix}\).

So, \(A = PDp^{-1} = A^{20} = PD^{20}p^{-1}\).

\[
\begin{aligned}
&: x_{20} = A^{20} x_0 = P D^{20} P^{-1} \begin{bmatrix} 3u \\ 3v \end{bmatrix} \\
&= \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}^{20} \begin{bmatrix} 3u \\ 3v \end{bmatrix} \\
&= \begin{bmatrix} 1 & -\left(\frac{1}{2}\right)^{10} \\ 2 & \left(\frac{1}{2}\right)^{10} \end{bmatrix} \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ -\frac{2}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 3u \\ 3v \end{bmatrix} \\
&= \begin{bmatrix} \frac{3}{12} + \frac{6}{12} \left(\frac{17}{20}\right)^{10} + \frac{1}{12} - \frac{1}{12} \left(\frac{17}{20}\right)^{20} \\ \frac{6}{12} - \frac{6}{12} \left(\frac{17}{20}\right)^{20} + \frac{2}{12} + \frac{1}{12} \left(\frac{17}{20}\right)^{20} \end{bmatrix} \begin{bmatrix} 3u \\ 3v \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{2} + \frac{5}{12} \left(\frac{17}{20}\right)^{20} - \frac{2}{12} \left(\frac{17}{20}\right)^{20} \\ \frac{3}{2} + \frac{5}{12} \left(\frac{17}{20}\right)^{20} \end{bmatrix} \begin{bmatrix} 3u \\ 3v \end{bmatrix} \\
&\approx \begin{bmatrix} 400 + \left(\frac{1}{2} - \frac{2}{12} \right)^{20} \\ \frac{3}{2} + \frac{5}{12} \left(\frac{17}{20}\right)^{20} \end{bmatrix}
\end{aligned}
\]

\[\text{will be in the city after 20 years.}\]
2. Express \( \frac{1+2i}{3-4i} + \frac{2-i}{5-i} \) as a real number.

\[
\frac{1+2i}{3-4i} + \frac{2-i}{5-i} = \frac{(3+4i)(1+2i) + (5-i)(2-i)}{(3-4i)(5-i)}
\]

\[
= \frac{3+10i+8i^2 + 10-5i}{9-16i^2} = \frac{3+10i-8}{9+16} + \frac{10-5}{25}
\]

\[
= \frac{-5+10i-10-5}{25} = \frac{-10}{25} = -\frac{2}{5}
\]

3. Consider \( z = \frac{i}{-2-3i} \).

a) Express \( z \) in rectangular form (i.e., write as \( z = a+ib \)).

\[
\frac{i}{-2-3i} = \frac{-2i-3i}{4+9} = \frac{-2-3i}{8}
\]

b) Express \( z \) in polar form.

\[
z = r \left( \cos \theta + i \sin \theta \right)
\]

\[
r = |z| = \sqrt{\left(\frac{-1}{8}\right)^2 + \left(\frac{-3}{8}\right)^2} = \sqrt{\frac{1}{64} + \frac{9}{64}} = \sqrt{\frac{10}{64}} = \sqrt{\frac{5}{32}} = \frac{\sqrt{5}}{4}
\]

\[
z = -\frac{1}{8} - \frac{3}{8}i = \frac{\sqrt{5}}{4} \left( \frac{-1}{2} - \frac{3}{2}i \right) = \frac{\sqrt{5}}{4} \left( \cos \frac{5\pi}{4} + i \sin \frac{5\pi}{4} \right)
\]

= \frac{\sqrt{5}}{4} e^{i \frac{5\pi}{4}}.
What is $\text{Arg}z$?

Recall: $\text{arg} z = \theta + 2\pi k$, $k \in \mathbb{Z}$ (multivalued), but $-\pi < \text{Arg} z \leq \pi$. Principal argument

$\frac{5\pi}{4} > \pi$, $\frac{5\pi}{4} - 2\pi = \frac{5\pi}{4} - \frac{8\pi}{4} = -\frac{3\pi}{4}$. $-\pi < -\frac{3\pi}{4} \leq \pi$.

So, $\text{Arg} z = -\frac{3\pi}{4}$.

What is $\overline{z}$?

Recall: If $z = a + bi$, then $\overline{z} = a - bi$.

So, $\overline{z} = -\frac{1}{4} + \frac{1}{4}i$.

(Or, in polar form we know that if $z = r e^{i\theta}$, then $\overline{z} = r e^{-i\theta}$, since $r e^{i\theta} = r (\cos \theta + i \sin \theta)$ $\overline{z} = r (\cos \theta - i \sin \theta) = r \cos \theta - i r \sin \theta = \overline{z}$.)

So, in polar form, $\overline{z} = \frac{\sqrt{2}}{4} e^{\frac{-\pi}{4} i}$.

4. Express $(\sqrt{3} - i)^6$ in polar form.

First, let’s express $\sqrt{3} - i$ in polar form.

$\Gamma = \sqrt{((\sqrt{3})^2) + (-1)^2} = \sqrt{4} = 2$.

So, $\sqrt{3} - i = 2 \left( \frac{\sqrt{3}}{2} - \frac{1}{2}i \right) = 2 \left( \cos \left( \frac{\pi}{6} \right) + i \sin \left( \frac{\pi}{6} \right) \right) = 2 e^{\frac{\pi}{6} i}$.
So we want to express \((2e^{i\pi})^6\) in polar form.

\[(2e^{i\pi})^6 = 2^6e^{6i\pi} = 64e^{6i\pi}\]

Since \(e^{i\pi} = e^{2\pi i} = -1\),

\[
\cos(1\pi) = \cos(2\pi i) = 1,
\sin(1\pi) = \sin(2\pi i) = 0.
\]

5. Find the solutions to the equation \(z^3 = -1\).

Recall: \(z = \sqrt[3]{-1} = \sqrt[3]{\cos(\pi + 2k\pi) + i\sin(\pi + 2k\pi)}\),

\[K = 0, 1, \ldots, n-1.\]

First let's write \(-1\) in polar form:

\[r = |\overline{-1}| = |-1|^2 = 1, \quad \theta = \pi.\]

So, \(-1 = \cos \pi + i\sin \pi = e^{i\pi}\).

Now \(z^3 = e^{i\pi}\). Let \(z = r_0 e^{i\phi}\).

So, \(r_0^3 e^{3i\phi} = e^{i\pi}\) + \(r_0^3 = 1\) + \(r_0 = 1\) + \(3\phi = \pi + 2k\pi\) + \(\phi = \frac{\pi}{3} + \frac{2k\pi}{3}\)

For \(K = 0, 1, 2\).

So, \(\phi = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{5\pi}{3} = \pi, \frac{3\pi}{2}, \frac{5\pi}{3}\).

So, \(z = e^{i\pi}, e^{i3\pi}, e^{i5\pi}\).

6. a) Find the square roots of \(2i\).

\[z = 2i = 2\left(\cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)\right) = 2e^{i\pi/2}\]

So, we want to solve \(z = \sqrt{2e^{i\pi/2}}\).

Let \(z = r_0 e^{i\phi}\).

So, \(r_0^2 e^{2i\phi} = 2e^{i\pi/2}\) + \(r_0^2 = 2\) \quad \rightarrow \quad r_0 = \sqrt{2}.

\[\phi = \frac{\pi}{4} + 2k\pi\] + \(\phi = \frac{3\pi}{4} + k\pi\) for \(K = 0, 1, 2\).

So, \(\phi = \frac{\pi}{4} + \frac{3\pi}{4}\).
So, the square roots of $2i$ are $\sqrt{2e^{\frac{3\pi i}{4}}}$ and $\sqrt{2e^{\frac{5\pi i}{4}}}$.  

(6) Express your two roots in rectangular coordinates.  

$\sqrt{2e^{\frac{3\pi i}{4}}} = \sqrt{2} (\cos(\frac{3\pi}{4}) + i \sin(\frac{3\pi}{4}))$  

$= \sqrt{2} (\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}) = 1 + i$.  

$\sqrt{2e^{\frac{5\pi i}{4}}} = \sqrt{2} (\cos(\frac{5\pi}{4}) + i \sin(\frac{5\pi}{4}))$  

$= \sqrt{2} (\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}) = 1 - i$.  