1. Suppose \( x_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \ x_2 = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \). Find an orthogonal basis for \( \text{Span} \ x_1, x_2 \).

Recall: To convert a basis \( \mathbf{v}_1, \ldots, \mathbf{v}_3 \) to an orthonormal basis \( \mathbf{u}_1, \ldots, \mathbf{u}_3 \), perform the following computations:

- \( \mathbf{v}_1 = \mathbf{u}_1 \).
- \( \mathbf{v}_2 = \mathbf{u}_2 - \text{proj}_{\mathbf{u}_1} \mathbf{v}_2 \).
- \( \mathbf{v}_3 = \mathbf{u}_3 - \text{proj}_{\mathbf{u}_1} \mathbf{v}_3 - \text{proj}_{\mathbf{u}_2} \mathbf{v}_3 \).
- \( \mathbf{v}_4 = \mathbf{u}_4 - \text{proj}_{\mathbf{u}_1} \mathbf{v}_4 - \text{proj}_{\mathbf{u}_2} \mathbf{v}_4 - \text{proj}_{\mathbf{u}_3} \mathbf{v}_4 \), etc.

- If \( \mathbf{S} = \mathbf{v}_1, \ldots, \mathbf{v}_3 \) is a set of orthogonal vectors, then \( \mathbf{S} \) is linearly independent.

We can see that \( x_1 \) and \( x_2 \) are not multiples of each other, so they are linearly independent \( \Rightarrow \) they form a basis for \( \text{Span} \ x_1, x_2 \). So, we want to use Gram-Schmidt to orthogonalize these vectors.

\[
\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.
\]

\[
\mathbf{v}_2 = \begin{pmatrix} 2 \\ 3 \end{pmatrix} - \text{proj}_{\mathbf{v}_1} \mathbf{v}_2 = \begin{pmatrix} 2 \\ 3 \end{pmatrix} - \frac{\mathbf{v}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1} \mathbf{v}_1 = \begin{pmatrix} 2 \\ 3 \end{pmatrix} - \frac{4}{11} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}.
\]

\[
\text{Check: } \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 3 \end{pmatrix} = 0.
\]

\[
\text{So, } \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \end{pmatrix} \right\} \text{ form an orthogonal basis for } \text{Span} \ x_1, x_2.
\]
b. Find an orthonormal basis for \( \text{span} \mathbf{v}_1, \mathbf{v}_3 \).

Recall: An orthogonal set in which each vector has norm 1 is said to be orthonormal.

So, we must normalize \((\mathbf{v}_1 + \mathbf{v}_3)\).

\[
\|\mathbf{v}_1\| = \sqrt{2}, \quad \|\mathbf{v}_3\| = \sqrt{3}.
\]

So, \[
\left( \frac{\sqrt{2}}{\sqrt{3}}, 0, 0 \right), \quad \left( 0, \frac{\sqrt{3}}{\sqrt{3}}, 1 \right)
\] forms an orthonormal basis for \( \text{span} \mathbf{v}_1, \mathbf{v}_3 \).

Check \[
\| \left( \frac{\sqrt{2}}{\sqrt{3}}, 0, 0 \right) \| = \sqrt{\frac{2}{3} + \frac{1}{3}} = 1, \quad \| \left( 0, \frac{\sqrt{3}}{\sqrt{3}}, 1 \right) \| = \sqrt{1} = 1.
\]

2. Suppose \( \mathbf{v}_1 = (1, 0, 0) \), \( \mathbf{v}_2 = \left( \frac{-1}{\sqrt{2}}, 1, 0 \right) \), \( \mathbf{v}_3 = \left( \frac{1}{\sqrt{2}}, 0, 1 \right) \).

\( \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \) forms a basis for a subspace of \( \mathbb{R}^3 \). Find an orthonormal basis for this subspace.

Again, we will use Gram-Schmidt to normalize:

\[
\mathbf{v}_1 = (1, 1, 1), \quad \mathbf{v}_2 = \mathbf{v}_2 - \text{Proj}_{\mathbf{v}_1} \mathbf{v}_2 = \left( 0, \frac{-1}{\sqrt{2}}, 0 \right), \quad \text{Proj}_{\mathbf{v}_1} \mathbf{v}_2 = \frac{1}{3} \mathbf{v}_1 = \left( \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \right),
\]

\[
\mathbf{v}_3 = \mathbf{v}_3 - \text{Proj}_{\mathbf{v}_1} \mathbf{v}_3 - \text{Proj}_{\mathbf{v}_2} \mathbf{v}_3 = \left( \frac{-1}{\sqrt{2}}, 0, 1 \right) - \left( 0, \frac{-1}{\sqrt{2}}, 0 \right) - \left( \frac{-1}{\sqrt{2}}, 0, 1 \right) = \left( 0, \frac{1}{\sqrt{2}}, 0 \right).
\]
\[
\begin{align*}
\left(\begin{array}{c}
4 \\
-2 \\
0
\end{array}\right) - \frac{4}{3} \left(\begin{array}{c}
1 \\
1 \\
1
\end{array}\right) + \frac{12}{18} \left(\begin{array}{c}
-3 \\
2/3 \\
-1
\end{array}\right) &= \left(\begin{array}{c}
-3 \\
1/3 \\
-1
\end{array}\right) \\
\left(\begin{array}{c}
1/3 \\
-5/3 \\
-5/3
\end{array}\right). & \text{Forms an orthogonal basis. Now we need to normalize:}
\end{align*}
\]

\[
\|v_1\| = \sqrt{4} = 2, \quad \|v_2\| = \sqrt{9 + 4 + 4 + 1} = \sqrt{18} = \sqrt{9 \cdot 2} = 3\sqrt{2}, \quad \|v_3\| = \sqrt{1 + \frac{25}{9} + \frac{49}{9} + \frac{25}{9}} = \sqrt{\frac{108}{9}} = \sqrt{\frac{4 \cdot 27}{9}} = \sqrt{4 \cdot 3} = 2\sqrt{3}.
\]

\[
\left(\begin{array}{c}
\frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{3}} \\
\frac{1}{\sqrt{6}}
\end{array}\right), \quad \left(\begin{array}{c}
\frac{-1}{\sqrt{2}} \\
\frac{2}{\sqrt{3}} \\
\frac{1}{\sqrt{6}}
\end{array}\right), \quad \left(\begin{array}{c}
\frac{1}{\sqrt{3}} \\
\frac{-5}{2} \\
\frac{-5}{2}\sqrt{3}
\end{array}\right) & \text{Forms an orthonormal basis.}
\]

\[
\left(\begin{array}{c}
-\frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{3}} \\
\frac{1}{\sqrt{6}}
\end{array}\right)
\]

3. (See #3 From Tutorial #11 Notes).

4. Suppose $x_1, x_2, x_3$ are linearly independent vectors in $\mathbb{R}$. Let $W = \text{span}\{x_1, x_2, x_3\}$.

\(\circ\) What is $\dim(W)$ (i.e., the dimension of $W$)?

Recall: All bases of a finite dim'l vector space have the same number of vectors.
• If a finite dimensional vector space \( V \) has a basis consisting of \( n \) vectors, then by definition \( \text{dim}(V) = n \).

So, since \( x_1, x_2, x_3 \) are linearly independent and they span \( W = \text{span}\{x_1, x_2, x_3\} \) is a basis for \( W \) \( \Rightarrow \) \( \text{dim}(W) = 3 \).

(b) Let \( x_4 \in W \). Is the set \( Y = \{x_1, x_2, x_3, x_4\} \) linearly independent?

Recall: Let \( \{v_1, \ldots, v_n\} \) be a basis for \( V \). Then:

- If \( S \subseteq V \) has \( n \) vectors \( \Rightarrow \) \( S \) linearly dependent.
- If \( S \subseteq V \) has \( < n \) vectors \( \Rightarrow \) \( S \) does not span \( V \).

So, since \( \{x_1, x_2, x_3\} \) is a basis for \( W \), \( Y \subseteq W \), but \( Y \) has \( 4 \) \( > \) \( 3 \) vectors \( \Rightarrow \) \( Y \) is not linearly independent.

(c) Let \( x_5, x_6 \in W \). Does \( \text{span}\{x_5, x_6\} = W \)?

\[ \text{span}\{x_5, x_6\} \neq W \text{ but } 2 < 3 \Rightarrow \text{span}\{x_5, x_6\} \text{ does not span } W \Rightarrow \text{span}\{x_5, x_6\} \neq W. \]

(d) Which familiar vector space does \( W \) equal?

Recall: \( \text{let } V \text{ be a vector space s.t. } \text{dim}(V) = n \).

\[ \text{let } S = \{x_1, \ldots, x_3\} \text{ s.t. } x_1, \ldots, x_3 \in V \text{. Then, } S \text{ is a basis for } V \Rightarrow S \text{ is linearly independent or } S \text{ spans } V. \]

The standard basis for \( \mathbb{R}^3 \) is \( \{(1,0,0), (0,1,0), (0,0,1)\} \).
5. Since the standard basis for $\mathbb{R}^3$ has 3 elements $\Rightarrow \text{dim}(\mathbb{R}^3) = 3$.

Let $S = \{x_1, x_2, x_3\}$. We know $x_1, x_2, x_3 \in \mathbb{R}^3$.
We also know $x_1, x_2, x_3$ are 3 linearly independent vectors $\Rightarrow S$ is a basis for $\mathbb{R}^3 \Rightarrow \text{span} \{x_1, x_2, x_3\} = \mathbb{R}^3$.

$\Rightarrow \text{dim} S = \text{dim}(\mathbb{R}^3)$.

(We could have also used the theorem that says:

If $W$ is a subspace of $V$, then $W \subseteq V \Rightarrow \text{dim}(W) = \text{dim}(V)$.)

5. Suppose I gave you a homogeneous system of equations, you solved it, and found:

$x = 2s + t - 3r$
$y = 2t$
$z = t$
$w = s$
$u = r$.

a) What would be a basis for your solution space?

b) What is the dimension of this solution space?

We can rewrite this solution as:

$$
\begin{pmatrix}
x \\
y \\
z \\
w \\
u
\end{pmatrix} = 
\begin{pmatrix}
2 \\
0 \\
2 \\
1 \\
0
\end{pmatrix} s + 
\begin{pmatrix}
0 \\
2 \\
1 \\
0 \\
0
\end{pmatrix} t + 
\begin{pmatrix}
-3 \\
0 \\
0 \\
0 \\
1
\end{pmatrix} r.
$$

So, $\{2, 2, -3\}$ is a basis for this homogeneous system (aka, basis for the nullspace).

The dimension of this solution space is 3.