Tutorial Info:

- **Website:** http://ms.mcmaster.ca/~dedieula.

- **Exam Review:** I’ll be doing an exam review Mon. Apr. 14th, 2:30-4:30pm in BSB147. (There are also 2 other reviews happening that day. See Avenue for more details.)

- **Math Help Centre:** Wednesdays 2:30-5:30pm.

- **Email:** dedieula@math.mcmaster.ca.
Examples:

1. a) Suppose $x_1 = (1, 1, 0)$ and $x_2 = (2, 2, 3)$. Find an orthogonal basis for $\text{span}\{x_1, x_2\}$.
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Recall: A set $S = \{v_1, \ldots, v_n\}$ of vectors, where $v_1, \ldots, v_n \in V$ is called a **basis** for $V$ if:

- The vectors in $S$ are linearly independent.
- $S$ spans $V$. 

Gram-Schmidt Process:

To convert a basis $\{u_1, \ldots, u_n\}$ to an orthogonal basis $\{v_1, \ldots, v_n\}$, perform the following computations:

1. $v_1 = u_1$
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b) Find an orthonormal basis for \( \text{span}\{x_1, x_2\} \).

**Recall:** A set of vectors is called **orthonormal** if it is orthogonal and each vector has norm 1.
Examples:

2. Suppose \( x_1 = (1, 1, 1, 1), \ x_2 = (-1, 4, 4, 1), \ x_3 = (4, -2, 2, 0), \) and \( \{x_1, x_2, x_3\} \) forms a basis for a subspace of \( \mathbb{R}^4 \). Find an orthonormal basis for this subspace.
Examples:

3. We know

\[ T := \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}. \]

forms a basis for \( \mathbb{R}^2 \).
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- a) Find the coordinate vector of \( \mathbf{v} = (3, 5) \) relative to the basis \( T \). i.e. Find \([\mathbf{v}]_T\).
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- **a)** Find the coordinate vector of \( v = (3,5) \) relative to the basis \( T \). i.e. Find \([v]_T\).

- **Recall:** If \( S = \{v_1, \ldots, v_n\} \) is a basis for \( V \), and \( w = k_1 v_1 + k_2 v_2 + \ldots + k_n v_n \) for \( k_1, \ldots, k_n \in \mathbb{R} \), then \([w]_S = (k_1, \ldots, k_n)\) is called the coordinate vector of \( v \) relative to \( S \).
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- b) Find the vector \( w \in \mathbb{R}^2 \) whose coordinate vector relative to \( T \) is \([w]_T = (4, 2)\).
Examples:

4. Suppose $x_1$, $x_2$, and $x_3$ are linearly independent vectors in $\mathbb{R}^3$. Let $W = \text{span}\{x_1, x_2, x_3\}$. 
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- a) What is $\text{dim}(W)$, (i.e. the dimension of $W$)?
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- **Recall:** All bases of a finite dimensional vector space $V$ have the same number of vectors.
Examples:

- 4. Suppose $x_1$, $x_2$, and $x_3$ are linearly independent vectors in $\mathbb{R}^3$. Let $W = \text{span}\{x_1, x_2, x_3\}$.

- a) What is $\dim(W)$, (i.e. the dimension of $W$)?

- Recall: All bases of a finite dimensional vector space $V$ have the same number of vectors.

- If a finite dimensional vector space $V$ has a basis consisting of $n$ vectors, then by definition, $\dim(V) = n$. 
Examples:

- b) Let $x_4 \in W$. Is the set $Y = \{x_1, x_2, x_3, x_4\}$ linearly independent?
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- Recall: Let $\{v_1, \ldots, v_n\}$ be a basis for $V$. Let $S$ be a set of vectors from $V$. Then:
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- **b)** Let \( x_4 \in W \). Is the set \( Y = \{x_1, x_2, x_3, x_4\} \) linearly independent?

- **Recall:** Let \( \{v_1, \ldots, v_n\} \) be a basis for \( V \). Let \( S \) be a set of vectors from \( V \). Then:
  1. If \( S \) has \( > n \) vectors, then \( S \) is linearly dependent.
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- Recall: Let $\{v_1, \ldots, v_n\}$ be a basis for $V$. Let $S$ be a set of vectors from $V$. Then:
  1. If $S$ has $> n$ vectors, then $S$ is linearly dependent.
  2. If $S$ has $< n$ vectors, then $S$ does not span $V$. 
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■ b) Let \( x_4 \in W \). Is the set \( Y = \{x_1, x_2, x_3, x_4\} \) linearly independent?

■ Recall: Let \( \{v_1, \ldots, v_n\} \) be a basis for \( V \). Let \( S \) be a set of vectors from \( V \). Then:
  1. If \( S \) has \( > n \) vectors, then \( S \) is linearly dependent.
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■ c) Let \( x_5, x_6 \in W \). Does \( \text{span}\{x_5, x_6\} = W \)?
Examples:

- d) Which familiar vector space is equal to $W$?
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- Recall: Let $V$ be a vector space such that $\dim(V) = n$. Let $S = \{x_1, \ldots, x_n\}$ be a set of vectors in $V$. Then, $S$ is a basis for $V \iff S$ is linearly independent OR $S$ spans $V$. 
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- d) Which familiar vector space is equal to $W$?

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- The **standard basis** for $\mathbb{R}^3$ is $\{(1,0,0), (0,1,0), (0,0,1)\}$.
Examples:

- 5. Suppose you were given a homogeneous linear system, you solved it, and found solutions: $x = 2s + t - 3r$, $y = 2t$, $z = t$, $w = s$, $u = r$. 
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- **5.** Suppose you were given a homogeneous linear system, you solved it, and found solutions: \( x = 2s + t - 3r \), \( y = 2t \), \( z = t \), \( w = s \), \( u = r \).

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- 5. Suppose you were given a homogeneous linear system, you solved it, and found solutions: \( x = 2s + t - 3r, y = 2t, z = t, w = s, u = r. \)

- a) Find a basis for this solution space.

- b) What is the dimension of this solution space?
Examples:

- Suppose $A$ is a $3 \times 4$ matrix. Complete the following sentences.
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  a) The rank of \( A \) is at most . . . .
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- **Recall:** The **rowspace** of $A$ is the subspace spanned by the rows of $A$.
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\[ \text{rank}(A) = \dim \text{rowspace of } A = \dim \text{columnspace of } A. \]

\[ \text{nullity}(A) = \dim \text{nullspace of } A. \]

**Rank-Nullity Theorem:** \[ \text{rank}(A) + \text{nullity}(A) = n, \] where $A$ is a $m \times n$ matrix.
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  b) Are the columns of $A$ linearly dependent?
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8. Suppose $A$ is a $3 \times 3$ matrix whose nullspace is a line through the origin in $\mathbb{R}^3$. Can the row or column space of $A$ be a line through the origin too?