Tutorial Info:

- **Website:** http://ms.mcmaster.ca/~dedieula.
- **Math Help Centre:** Wednesdays 2:30-5:30pm.
- **Email:** dedieula@math.mcmaster.ca.
Elementary Matrices

- An elementary matrix is a $n \times n$ matrix that can be obtained from the identity $I_n$ by performing a single elementary row operation.

\[ \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \] is an elementary matrix that corresponds to the row operation $r_2 \leftarrow r_2 + r_1$.

So, when we do a row operation to a $n \times n$ matrix $A$, this is equivalent to multiplying $A$ by an elementary matrix.

\[ A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \]

$r_2 \leftarrow r_2 + r_1 = \begin{pmatrix} 1 & 0 \\ 4 & 6 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$. 
Elementary Matrices

- An elementary matrix is a $n \times n$ matrix that can be obtained from the identity $I_n$ by performing a single elementary row operation.

- e.g.

$$E_1 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

is an elementary matrix that corresponds to the row operation $r_2 \leftarrow r_2 + r_1$. 
Elementary Matrices

- An **elementary matrix** is a $n \times n$ matrix that can be obtained from the identity $I_n$ by performing a single elementary row operation.

- **e.g.**

  $$E_1 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

  is an elementary matrix that corresponds to the row operation $r_2 \leftarrow r_2 + r_1$.

- So, when we do a row operation to a $n \times n$ matrix $A$, this is equivalent to multiplying $A$ by an elementary matrix. **e.g.**

  $$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \quad \overset{r_2 \leftarrow r_2 + r_1}{\rightarrow} \quad \begin{pmatrix} 1 & 2 \\ 4 & 6 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}.$$
Examples:

1.a) Consider

\[ A = \begin{pmatrix} 2 & -4 \\ -2 & 3 \end{pmatrix}. \]

Write \( A \) as a product of elementary matrices.
Examples:

- 1.a) Consider

\[ A = \begin{pmatrix} 2 & -4 \\ -2 & 3 \end{pmatrix}. \]

Write \( A \) as a product of elementary matrices.

- **Recall:** To do this we should:
Examples:

1.a) Consider

\[ A = \begin{pmatrix} 2 & -4 \\ -2 & 3 \end{pmatrix}. \]

Write \( A \) as a product of elementary matrices.

Recall: To do this we should:

1. Reduce \( A \) to the identity \( I \).
1.a) Consider

\[ A = \begin{pmatrix} 2 & -4  \\ -2 & 3 \end{pmatrix}. \]

Write \( A \) as a product of elementary matrices.

Recall: To do this we should:
1. Reduce \( A \) to the identity \( I \).
2. Keep track of row operations.
Examples:

1.a) Consider

\[ A = \begin{pmatrix} 2 & -4 \\ -2 & 3 \end{pmatrix}. \]

Write \( A \) as a product of elementary matrices.

**Recall:** To do this we should:

1. Reduce \( A \) to the identity \( I \).
2. Keep track of row operations.
3. Write each row operation as an elementary matrix.
Examples:

1.a) Consider

\[ A = \begin{pmatrix} 2 & -4 \\ -2 & 3 \end{pmatrix}. \]

Write \( A \) as a product of elementary matrices.

Recall: To do this we should:

1. Reduce \( A \) to the identity \( I \).
2. Keep track of row operations.
3. Write each row operation as an elementary matrix.
4. Express the row reduction as matrix multiplication.
Examples:

1.a) Consider

\[ A = \begin{pmatrix} 2 & -4 \\ -2 & 3 \end{pmatrix}. \]

Write \( A \) as a product of elementary matrices.

Recall: To do this we should:

1. Reduce \( A \) to the identity \( I \).
2. Keep track of row operations.
3. Write each row operation as an elementary matrix.
4. Express the row reduction as matrix multiplication.
5. Solve for \( A \).
Examples:

- b) Is this decomposition of $A$ into elementary matrices unique?
Examples:

- c) Find $A^{-1}$ without using the formula

$$
\frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.
$$
Examples:

- **Note:** Our work in Question 1 demonstrates why the inverse algorithm works.
Examples:

- **Note:** Our work in Question 1 demonstrates why the inverse algorithm works.
- **Inverse Algorithm:** To find the inverse of an invertible matrix $A$:
Examples:

- **Note:** Our work in Question 1 demonstrates why the inverse algorithm works.
- **Inverse Algorithm:** To find the inverse of an invertible matrix $A$:
  1. Find a sequence of elementary row operations that reduce $A$ to $I_n$. 
Examples:

- **Note:** Our work in Question 1 demonstrates why the inverse algorithm works.
- **Inverse Algorithm:** To find the inverse of an invertible matrix $A$:
  1. Find a sequence of elementary row operations that reduce $A$ to $I_n$.
  2. Perform those same row operations on $I_n$ to obtain $A^{-1}$.
Examples:

- **Note:** Our work in Question 1 demonstrates why the inverse algorithm works.
- **Inverse Algorithm:** To find the inverse of an invertible matrix $A$:
  1. Find a sequence of elementary row operations that reduce $A$ to $I_n$.
  2. Perform those same row operations on $I_n$ to obtain $A^{-1}$.
- **i.e.** These row operations can be written as elementary matrices: $E_k \ldots E_2 E_1 A = I$
Examples:

- **Note:** Our work in Question 1 demonstrates why the inverse algorithm works.

- **Inverse Algorithm:** To find the inverse of an invertible matrix $A$:
  1. Find a sequence of elementary row operations that reduce $A$ to $I_n$.
  2. Perform those same row operations on $I_n$ to obtain $A^{-1}$.

- **i.e.** These row operations can be written as elementary matrices: $E_k \ldots E_2 E_1 A = I$
Examples:

- **Note:** Our work in Question 1 demonstrates why the inverse algorithm works.

- **Inverse Algorithm:** To find the inverse of an invertible matrix $A$:
  1. Find a sequence of elementary row operations that reduce $A$ to $I_n$.
  2. Perform those same row operations on $I_n$ to obtain $A^{-1}$.

- **i.e.** These row operations can be written as elementary matrices: $E_k \ldots E_2 E_1 A = I$
  \[ \Rightarrow A^{-1} = E_k \ldots E_2 E_1. \]
Examples:

- **Note:** Our work in Question 1 demonstrates why the inverse algorithm works.
- **Inverse Algorithm:** To find the inverse of an invertible matrix $A$:
  1. Find a sequence of elementary row operations that reduce $A$ to $I_n$.
  2. Perform those same row operations on $I_n$ to obtain $A^{-1}$.
- **i.e.** These row operations can be written as elementary matrices: $E_k \ldots E_2 E_1 A = I$ 
  $\Rightarrow A^{-1} = E_k \ldots E_2 E_1$.
- So, to do this quickly, we perform the row operations represented by $E_k \ldots E_1$ simultaneously to $A$ and $I_n$ by adjoining $A$ with $I_n$: $[A|I_n] \rightarrow [I_n|A^{-1}]$. 
Examples:

2. Consider

\[ A = \begin{pmatrix} 1 & 1 & 1 \\ 6 & 7 & 5 \\ 3 & 2 & 3 \end{pmatrix}. \]
Examples:

2. Consider

\[ A = \begin{pmatrix} 1 & 1 & 1 \\ 6 & 7 & 5 \\ 3 & 2 & 3 \end{pmatrix}. \]

Using row operations we could find

\[ A^{-1} = \begin{pmatrix} -11 & 1 & 2 \\ 3 & 0 & 1 \\ 9 & -1 & -1 \end{pmatrix}. \]
Examples:

- a) Does

\[ Ax = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \]

have a unique solution?
Examples:

- a) Does

\[
Ax = \begin{pmatrix}
1 \\
2 \\
3
\end{pmatrix}
\]

have a unique solution?

- **Recall**: We know several equivalent statements, where \( A \) is a \( n \times n \) matrix:
Examples:

- a) Does

\[ Ax = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \]

have a unique solution?

- **Recall:** We know several equivalent statements, where \( A \) is a \( n \times n \) matrix:

  (a) \( A \) is invertible.
Examples:

- a) Does

\[ Ax = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \]

have a unique solution?

- Recall: We know several equivalent statements, where \( A \) is a \( n \times n \) matrix:
  
  (a) \( A \) is invertible.
  
  (b) \( Ax = 0 \) has only the trivial solution.
Examples:

- a) Does

\[ Ax = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \]

have a unique solution?

- Recall: We know several equivalent statements, where \( A \) is a \( n \times n \) matrix:
  
  (a) \( A \) is invertible.
  
  (b) \( Ax = 0 \) has only the trivial solution.
  
  (c) The reduced row echelon form of \( A \) is \( I_n \).
Examples:

- a) Does

\[ Ax = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \]

have a unique solution?

- Recall: We know several equivalent statements, where \( A \) is a \( n \times n \) matrix:
  
  (a) \( A \) is invertible.
  
  (b) \( Ax = 0 \) has only the trivial solution.
  
  (c) The reduced row echelon form of \( A \) is \( I_n \).
  
  (d) \( A \) is expressible as the product of elementary matrices.
Examples:

- **a)** Does

\[ Ax = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \]

have a unique solution?

- **Recall:** We know several equivalent statements, where \( A \) is a \( n \times n \) matrix:
  
  (a) \( A \) is invertible.
  
  (b) \( Ax = 0 \) has only the trivial solution.
  
  (c) The reduced row echelon form of \( A \) is \( I_n \).
  
  (d) \( A \) is expressible as the product of elementary matrices.
  
  (e) \( Ax = b \) is consistent for every \( n \times 1 \) matrix \( b \).
Examples:

- **a)** Does

\[
Ax = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}
\]

have a unique solution?

- **Recall:** We know several equivalent statements, where \( A \) is a \( n \times n \) matrix:
  
  (a) \( A \) is invertible.
  
  (b) \( Ax = 0 \) has only the trivial solution.
  
  (c) The reduced row echelon form of \( A \) is \( I_n \).
  
  (d) \( A \) is expressible as the product of elementary matrices.
  
  (e) \( Ax = b \) is consistent for every \( n \times 1 \) matrix \( b \).
  
  (f) \( Ax = b \) has exactly one solution for every \( n \times 1 \) matrix \( b \).
Examples:

- b) Solve for $x$. 
Examples:

- 3. Consider

\[ A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 7 & 1 \end{pmatrix} \]
Examples:

- 3. Consider

\[
A = \begin{pmatrix}
1 & 2 & 3 \\
2 & 4 & 6 \\
1 & 7 & 1 \\
\end{pmatrix}.
\]

- a) Is \( A \) invertible?
Examples:

3. Consider 

\[ A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 7 & 1 \end{pmatrix}. \]

a) Is \( A \) invertible?

b) Does \( Ax = 0 \) have nontrivial solutions?