Tutorial Info:

- **Website:** http://ms.mcmaster.ca/~dedieula.
- **Review Session:** I’ll be doing a review session Mon. March 24th, 6:30-8:30pm, HH302. (See Avenue to Learn for additional review sessions.)
- **Math Help Centre:** Wednesdays 2:30-5:30pm.
- **Email:** dedieula@math.mcmaster.ca.
Examples:

1. Let $V = \mathbb{R}^2$ and define addition and scalar multiplication as follows: If $u = (x_1, y_1), v = (x_2, y_2)$, then

$$u + v = \begin{pmatrix} x_1 - 2x_2 + 1 \\ 2y_1 + 3y_2 - 4 \end{pmatrix},$$

$$\alpha u = \begin{pmatrix} \frac{1}{\alpha}x_1 \\ y_1 \alpha^2 \end{pmatrix}.$$
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Is $V$ a vector space with these stated operations? Specify which axioms hold, and which fail.
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■ Recall: A vector space is a set $V$ together with a binary operation “$+$” and a rule for scalar multiplication satisfying 10 axioms.
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$$u + v = \left( \begin{array}{c} x_1 - 2x_2 + 1 \\ 2y_1 + 3y_2 - 4 \end{array} \right),$$

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Recall: A vector space is a set $V$ together with a binary operation “+” and a rule for scalar multiplication satisfying 10 axioms. i.e. If the axioms hold for all vectors $v, u, w \in V$ and for all scalars $\alpha, \beta \in \mathbb{R}$, then $V$ is a vector space.
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Note: Scalars do not have to be in $\mathbb{R}$, but for simplicity I’ll use $\mathbb{R}$ here.
Vector Space Axioms:

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5. “+” **Inverse**: For each $v \in V \exists (-v) \in V$ such that $v + (-v) = \vec{0}$. 
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6. “\( \alpha \)” Closure: \( v \in V \Rightarrow \alpha v \in V \ \forall \alpha \in \mathbb{R} \).
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6. “$\alpha$” Closure: $v \in V \Rightarrow \alpha v \in V \forall \alpha \in \mathbb{R}$.
7. “$\alpha$” Distributivity: $\alpha(v + w) = \alpha v + \alpha w \forall, w \in V, \alpha \in \mathbb{R}$.
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8. **Vector Distributivity:** \( (\alpha + \beta)v = \alpha v + \beta v \ \forall v \in V, \alpha, \beta \in \mathbb{R} \).
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8. **Vector Distributivity:** \( (\alpha + \beta)v = \alpha v + \beta v \forall v \in V, \alpha, \beta \in \mathbb{R} \).
9. “\( \alpha \)” **Associativity:** \( (\alpha (\beta v)) = (\alpha \beta) v \forall v \in V, \alpha, \beta \in \mathbb{R} \).
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10. “\( \alpha \)” Identity: \( 1 \times v = v \ \forall v \in V, \ 1 \in \mathbb{R} \).
Examples:

2. If $V = \mathbb{R}^2$ is a set with addition and scalar multiplication defined as $u + v = (u_1 + v_1 + 1, u_2 + v_2 + 1)$, $\alpha u = (\alpha u_1, \alpha u_2)$, where $u = (u_1, u_2)$, $v = (v_1, v_2)$, then what must $\tilde{0}$ be?
Examples:

3. Determine which of the following sets are subspaces of $P_2$ (where $P_2$ is the vector space of polynomials of degree $\leq 2$. e.g. $\{ax^2 + bx + c \mid a, b, c \in \mathbb{R}\}$).
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- **Recall:** A subset $W$ of a vector space $V$ is called a **subspace** of $V$ if $W$ is itself a vector space under the addition and multiplication operations defined on $V$. 
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Recall: A subset $W$ of a vector space $V$ is called a **subspace** of $V$ if $W$ is itself a vector space under the addition and multiplication operations defined on $V$.

Subspace Criterion: A subset $W \subseteq V$ is a subspace of $V \iff$ the following hold:
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- **Subspace Criterion:** A subset $W \subseteq V$ is a subspace of $V \iff$ the following hold:
  1. $W$ is nonempty.
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1. $W$ is nonempty.
2. $W$ is closed under addition (i.e. $u, v \in W \Rightarrow u + v \in W \forall$ scalars $\alpha$).
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a) $W = \{r(1 + x^2) \mid r \in \mathbb{R}\}$. 
Examples:

- **3.** Determine which of the following sets are subspaces of $P_2$ (where $P_2$ is the vector space of polynomials of degree $\leq 2$. e.g. $\{ax^2 + bx + c | a, b, c \in \mathbb{R}\}$).

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- **a)** $W = \{r(1 + x^2) | r \in \mathbb{R}\}$.

- **b)** $Y = \{\text{quadratic polynomials with only real roots}\}$.
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a) $W = \{r(1 + x^2) | r \in \mathbb{R}\}$.

b) $Y = \{\text{quadratic polynomials with only real roots}\}$.

c) $Z = \{a + bx | a, b \in \mathbb{R}, a^2 = b^2\}$. 
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c) $Z = \{a + bx | a, b \in \mathbb{R}, a^2 = b^2\}$.

d) $J = \{p + qx + rx^2 | p, q, r \in \mathbb{R}, r \geq 0\}$.
Examples:

- 4. Is the set $W_1 = \{(v_1, v_2, 0) | v_1, v_2 \in \mathbb{R}\}$ a subspace of $\mathbb{R}^3$?
Examples:

5. Consider the following sets of vectors:

\[ S_1 := \left\{ \begin{pmatrix} 9 \\ -4 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 8 \end{pmatrix} \right\}, S_2 := \left\{ \begin{pmatrix} 9 \\ -4 \\ 2 \end{pmatrix}, \begin{pmatrix} 4 \\ 6 \\ -3 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} \right\}, \]

\[ S_3 := \left\{ \begin{pmatrix} 9 \\ -4 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 8 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \right\}. \]
Examples:

- 5. Consider the following sets of vectors:

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\begin{align*}
S_1 &: = \left\{ \begin{pmatrix} 9 \\ -4 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 8 \end{pmatrix} \right\}, \\
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- a) Which sets span \( \mathbb{R}^3 \)?
Examples:

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a) Which sets span \( \mathbb{R}^3 \)?

Recall: The span of a set \( S = \{w_1, \ldots, w_r\} \), is the subspace formed by taking all possible linear combinations of the vectors in \( S \). i.e. \( \text{span}(S) = \{\alpha_1 w_1 + \ldots \alpha_r w_r | \alpha_1, \ldots, \alpha_r \in \mathbb{R}\} \).
Examples:

1. Consider the following sets of vectors:

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2. Which sets span \( \mathbb{R}^3 \)?

Recall: The span of a set \( S = \{w_1, \ldots, w_r\} \), is the subspace formed by taking all possible linear combinations of the vectors in \( S \). i.e.

\[ \text{span}(S) = \{ \alpha_1 w_1 + \ldots \alpha_r w_r | \alpha_1, \ldots, \alpha_r \in \mathbb{R} \}. \]

Recall: If \( A \) is square, then \( Ax = b \) is consistent for every \( n \times 1 \) matrix \( b \) \( \iff \det(A) \neq 0 \).
Examples:

- b) Is the vector

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\begin{pmatrix}
3 \\
-1 \\
2
\end{pmatrix}
\]
in the span of \( S_1 \)? \( S_2 \)? \( S_3 \)?
Examples:

- b) Is the vector
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  \end{pmatrix}
  \]
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- c) Which of these vectors are linearly independent?
Examples:

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- c) Which of these vectors are linearly independent?

**Recall:** If a set of vectors \( S = \{v_1, \ldots, v_r\} \) is such that the equation
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\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_r v_r = \vec{0}
\]
has only the trivial solution (i.e. \( \alpha_1 = \ldots = \alpha_r = 0 \)),
then these vectors are said to be **linearly independent**. If there exist nontrivial solutions, then the vectors are said to be **linearly dependent**.