Tutorial Info:

- **Website:** [http://ms.mcmaster.ca/~dedieula](http://ms.mcmaster.ca/~dedieula).
- **Math Help Centre:** Wednesdays 2:30-5:30pm.
- **Email:** dedieula@math.mcmaster.ca.
Examples:

1. Consider

\[ A = \begin{pmatrix} 3 & 2 & 4 \\ 1 & 1 & 2 \\ 1 & 5 & 3 \end{pmatrix}. \]

Recall: For a $2 \times 2$ matrix

\[ B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \]

\[ \det(B) = ad - bc. \]

Recall: If $D$ is a square, then the minor of entry $a_{ij}$, $M_{ij}$ is the determinant of the submatrix that remains after the $i^{th}$ row and $j^{th}$ column are deleted from $D$.

Cofactor of entry $a_{ij}$, $C_{ij}$: is $kM_{ij}$, where $k = 1$ or $-1$ in accordance with the pattern in the checkerboard array:

\[ B = \begin{pmatrix} + & - & + & - & + & \ldots \\ - & + & - & + & - & \ldots \\ + & - & + & - & + & \ldots \\ - & + & - & + & - & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}, \]
Examples:

- a) Find $M_{11}, M_{12}, M_{13}, C_{11}, C_{12},$ and $C_{13}$
- b) Find $\det(A)$.

Recall: You can find $\det(A)$ by multiplying the entries in any row or column by their corresponding cofactor and adding the resulting products.

Note: We could have chosen a different row or column.
Examples:

2. Consider

\[ A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 0 & 2 \end{pmatrix}. \]

Find \( \det(A) \).

Note: Choosing a row or column with lots of zeros makes things easier!
Examples:

2. Consider

\[
A = \begin{pmatrix}
1 & 2 & 3 \\
2 & 4 & 6 \\
3 & 5 & 7
\end{pmatrix}.
\]

a. Find det(A).

Recall: How do elementary row operations affect matrices?

Let B be a square matrix, and let C denote what B becomes after each row operation.

1. Multiply row by nonzero scalar k: \( \det(C) = k \det(B) \).
2. Switch any 2 rows: \( \det(C) = -\det(B) \).
3. Add a multiple of a row to an existing row: \( \det(C) = \det(B) \).

Doing the same operations on B’s columns yield the same results.
Examples:

b) Consider

\[ B = \begin{pmatrix} t & 2t & 3t \\ 2 & 4 & 6 \\ 3 & 5 & 7 \end{pmatrix}, \]

for \( t \in \mathbb{R} \). Find \( \det(B) \).
Examples:

4. Suppose \( \det(A) = 3 \), \( \det(B) = 9 \), \( \det(C) = 2 \). What is \( \det(X) \), if \( BX = 6C^TA \)?

Recall: We know the following properties concerning determinants:

(a) \( \det(A) = \det(A^T) \)
(b) \( \det(AB) = \det(A)\det(B) \)
(c) \( \det(kA) = k^n \det(A) \), where \( k \in \mathbb{R} \), and \( A \) is a \( n \times n \) matrix.
(d) \( \det(A) \neq 0 \iff A \) is invertible.
(e) \( \det(A^{-1}) = \frac{1}{\det(A)} \).
Examples:

■ 5.a) Consider

\[ A = \begin{pmatrix} 1 & x & 2 \\ 3 & 1 & -1 \\ -1 & 2 & 2 \end{pmatrix}. \]

When is \( A \) singular?

■ Recall: A matrix \( A \) is called singular if \( A \) is not invertible.

■ Also, we know that \( A \) invertible \( \iff \) \( \det(A) \neq 0 \), so \( A \) singular \( \iff \) \( \det(A) = 0 \).

■ So, we’re looking for the values of \( x \) such that \( \det(A) = 0 \).

■ b) When is \( A \) invertible?
Examples:

6. Consider

\[ A = \begin{pmatrix} 0 & 2 & 1 \\ -1 & -3 & 1 \\ -2 & -1 & -2 \end{pmatrix}. \]

Find \( A^{-1} \) using the adjoint method.

Recall: If \( A \) is invertible, then \( A^{-1} = \frac{1}{\det(A)} \text{adj}(A) \), where

\[
\text{adj}(A) = \begin{pmatrix} C_{11} & C_{12} & \ldots & C_{1n} \\ C_{21} & C_{22} & \ldots & C_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ C_{n1} & C_{n2} & \ldots & C_{nn} \end{pmatrix}^T. 
\]
Examples:

7. Solve the following linear system using Cramer’s Rule: \(3x + 2y = 1, 5x + 4y = -1\).

Cramer’s Rule: If \(Ax = b\) is a system of \(n\) linear equations in \(n\) unknowns such that \(\text{det}(A) \neq 0\), then \(Ax = b\) has a unique solution.

This solution is: \(x_1 = \frac{\text{det}(A_1)}{\text{det}(A)}, x_2 = \frac{\text{det}(A_2)}{\text{det}(A)}, \ldots, x_n = \frac{\text{det}(A_n)}{\text{det}(A)}\), where \(A_j\) is the matrix obtained by replacing the entries in the \(j\)th column of \(A\) by the entries in the matrix \(b\).