2.3: Linear Eq. is:

25, 27, 33, 46, 48

25. \( y' = x + 5y, \ y(0) = 3 \).

Recall: A first-order linear DE has the form

\[ a_1(x) \frac{dy}{dx} + a_0(x) y = g(x). \]

Its standard form is

\[ \frac{dy}{dx} + p(x) y = F(x). \]

To solve, we want to integrate both sides of

\[ \frac{dy}{dx} = e^{\int p(x) \, dx} \int e^{\int p(x) \, dx} \, F(x) \, dx \]

Here, putting our DE in standard form:

\[ \frac{dy}{dx} + (5)y = x \]

\[ e^{-5x} y = \int x \, e^{-5x} \, dx \]

\[ e^{-5x} y = \frac{-5x + 5}{5} \]

\[ y = -\frac{1}{5} x - \frac{1}{25} + c e^{5x} \]

\( y(0) = 3 \implies 3 = -\frac{1}{5} + c \implies c = \frac{16}{25} \).

\[ y = -\frac{1}{5} x - \frac{1}{25} + \frac{16}{25} e^{5x} \]
y is defined on (-∞, ∞).

\[ y' = -\frac{1}{5} + \frac{7}{5} e^{5x} \text{ cont. on } (-∞, ∞). \]

\[ \therefore I = (-∞, ∞). \]

27. \[ xy' + y = e^x; \ y(1) = a. \]

\[ \frac{dy}{dx} = \frac{e^x}{x} \]

\[ \int y \, dx = \int \frac{e^x}{x} \, dx = \ln |x| + C \]

\[ y(1) = a \]

\[ e^a \cdot y = \int e^x \left( \frac{e^x}{x} \right) \, dx \]

\[ \Rightarrow a = e + C \]

\[ \Rightarrow C = a - e \]

\[ xy = \int e^x \, dx \]

\[ x \cdot y = e^x + C \]

\[ y = xe^x + \frac{a - e}{x} \]

To find the largest interval where the solution is defined, we need the largest I x s.t. y is defined on I and its derivative is continuous.

y defined everywhere except at y = 0.

\[ y' = \ln(x) e^x + \frac{x e^x + (a - e) \ln(x)}{x} \text{ defined and cont. for } x > 0. \]

\[ \therefore I = (0, ∞). \]
33. \((x+1) \frac{dy}{dx} + y = \ln x \), \(y(1) = 10\). 

\[ \frac{dy}{dx} + \frac{y}{x+1} = \frac{\ln x}{x+1} \]

\[ \int \frac{dy}{dx} \, dx = \int \frac{\ln x}{x+1} \, dx \]

\[ y = \int e^x \, dx \]

\[ y = \frac{\ln x - x + c}{x+1} \]

\[ (x+1)y = \ln x - x + c \]

\[ y = \frac{\ln x - x}{x+1} + \frac{c}{x+1} \]

\[ \frac{dy}{dx} = \frac{\ln x (x+1)^{-1} + 2x (x+1)^{-1} - x (x+1)^{-1}}{(x+1)^2} \]

\[ = \frac{(x+1)^{-1} + x \ln x [-(x+1)^{-2}] + 2x (1)}{x+1} \]

\[ \text{Defined for } x > 0. \]

\[ \therefore \mathcal{I} = (0, 0). \]

Note: The solution is not really complete. You can just see that if you have \( \ln x \) and \( x+1 \)'s in the denominator.
46. Reread Example 6 and then find the general solution of the DE on the interval \((-3, 3)\).

\[ (x^2 - 9) \frac{dy}{dx} + xy = 0. \]

Recall: 
- Define \( u = x^2 - 9 \) and \( du = 2x\,dx \)
- \( \int 1\,dx = \int \frac{x}{x^2 - 9}\,dx = \frac{1}{2} \ln |x^2 - 9| + C \)

on the interval \((-3, 3)\), \( x^2 - 9 < 0 \) \( \Rightarrow \frac{1}{2} \ln |x^2 - 9| = \frac{1}{2} \ln (x^2 - 9) \)

\[ y = \int e^{\frac{x}{x^2 - 9}} (0) \]

\[ \Rightarrow \sqrt{9 - x^2} y = 0 + C \]

\[ \Rightarrow y = \frac{C}{\sqrt{9 - x^2}}. \]

48. Reread Example 6 and then discuss why it's technically incorrect to say that \( y = 5 \cdot e^{-x} \), \( 0 \leq x \leq 1 \) is a "solution" of the IVP on the interval \([0, \infty)\).

In order for this to be a "solution," we need \( y \) to be \( C' \) on \([0, \infty)\).

But notice that \( y \) is not differentiable at \( x = 1 \):

\[ \lim_{x \to 1^-} \frac{y(x) - y(1)}{x - 1} = -e^{-1}, \quad \lim_{x \to 1^+} \frac{y(x) - y(1)}{x - 1} = (e-1)(-e^{-1}) \]

\( \therefore \) at \( C' \) is not valid at \( x = 1 \), not same.