30. For what integers $p > 0$ is $F(x) = \begin{cases} x^p \sin \left( \frac{1}{x} \right) & x \neq 0 \\ 0 & x = 0 \end{cases}$ differentiable? For what $p$ is the derivative continuous?

\[ F'(x) = \begin{cases} px^{p-1} \sin \left( \frac{1}{x} \right) + x^p \cos \left( \frac{1}{x} \right) \cdot \left( 1 - \frac{1}{x^2} \right) & x \neq 0 \\ 0 & x = 0 \end{cases} \]

\[ = px^{p-1} \sin \left( \frac{1}{x} \right) - px^p \cos \left( \frac{1}{x} \right). \]

By definition a function is "differentiable" if it is differentiable at every point in its domain.

If $x \neq 0$ we can see that $F$ will be differentiable. So, let's see what happens at $x = 0$:

\[ F'(0) = \lim_{h \to 0} \frac{F(0+h) - F(0)}{h} = \lim_{h \to 0} \frac{h \sin \left( \frac{1}{h} \right) - 0}{h}. \]

As $h \to 0 \sin \left( \frac{1}{h} \right)$ keeps oscillating $b/w -1 \leq \sin \left( \frac{1}{h} \right) \leq 1$. So if $p = 1$ we can see $\lim_{h \to 0} \frac{h \sin \left( \frac{1}{h} \right)}{h}$ does not exist. However, if $p$ is an integer s.t. $p > 1 \Rightarrow p-1 > 0 \Rightarrow \lim_{h \to 0} \frac{h^p \sin \left( \frac{1}{h} \right)}{h} = \lim_{h \to 0} h^{p-1} \sin \left( \frac{1}{h} \right) = 0.

\[ \therefore F \text{ is differentiable for integers } p > 1. \]

If $x \neq 0$ we can see $F'(x)$ is continuous for all $p > 0$. So let's see what happens at $x = 0$:

We want to find the $p > 0$ s.t. $\lim_{x \to 0} px^{-1} \sin (x) - x \cos (x) = \ldots$
Again, \( \sin(\frac{1}{x}) + \cos(\frac{1}{x}) \) keep oscillating as \( x \to 0 \). So if \( p > 1 \), we can see that this limit won't exist: 
\[
\lim_{x \to 0} \sin(\frac{1}{x}) - x^p \cos(\frac{1}{x}) \text{ not exist.}
\]

\( p = 2 \) the limit does not exist: 
\[
\lim_{x \to 0} 2x \sin(\frac{1}{x}) - \cos(\frac{1}{x}) \text{ not exist.}
\]

However, if \( p > 2 \) is an integer, then 
\[
\lim_{x \to 0} p x^{p-1} \sin(\frac{1}{x}) - x^{p-2} \cos(\frac{1}{x}) = 0.
\]

So, the derivative is continuous for integers \( p > 2 \).

**Notice:** When \( p = 2 \) are function is:
\[
F(x) = \begin{cases} 
2x \sin(\frac{1}{x}) & x \neq 0 \\
0 & x = 0 
\end{cases}
\]

We just showed that this function is differentiable but its derivative is not continuous (i.e., its differentiable but not \( C' \) or not continuously differentiable).

So, we know continuous partials \( \Rightarrow \) differentiable, but differentiable \( \not\Rightarrow \) continuous partials. 

33. Let \( F: \mathbb{R}^4 \to \mathbb{R} \) & \( c(t): \mathbb{R} \to \mathbb{R}^4 \). Suppose, \( DF(1, 1, \pi, e^6) = (0, 1, 3, -7) \), \( c(T) = (1, 1, \pi, e^6) \) & \( c'(\pi) = (19, 11, 0, 1) \). Find \( \frac{d}{dt} F(c(t)) \) when \( t = \pi \).

\[
\frac{d}{dt} F(c(t)) = F'(c(t)) \cdot c'(\pi) = F'(1, 1, \pi, e^6) \cdot c'(\pi) = [0, 1, 3, -7] \begin{bmatrix} 19 \\ 11 \\ 0 \\ 1 \end{bmatrix} = 11 - 7 = 4.
\]
35. Let \( z = F(x - y) \). Use the chain rule to show that
\[
\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 0.
\]

\[
\frac{\partial z}{\partial x} = \frac{\partial F}{\partial g} \frac{\partial g}{\partial x} = \frac{\partial F}{\partial g} (1) = \frac{\partial F}{\partial g}.
\]

\[
\frac{\partial z}{\partial y} = \frac{\partial F}{\partial g} \frac{\partial g}{\partial y} = \frac{\partial F}{\partial g} (-1) = -\frac{\partial F}{\partial g}.
\]

So,
\[
\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 0.
\]

2.b.3. Compute the directional derivatives of the following along unit vectors at the indicated points in directions parallel to the given vector.

Recall: If \( F: \mathbb{R}^3 \to \mathbb{R} \), the directional derivative of \( F \) at \( x \) along the unit vector \( v \) is given by
\[
\nabla F \cdot v = \left. \frac{\partial F}{\partial t} F(x+tv) \right|_{t=0},
\]
if this exists.

\( F(x,y) = x^y \), \((x,y) = (e,e)\), \( \mathbf{d} = 5\mathbf{i} + 12\mathbf{j} \).

\[
d = 5\mathbf{i} + 12\mathbf{j} = (5,12) \quad \text{is not a unit vector since}
\]
\[
\sqrt{5^2 + 12^2} = \sqrt{25 + 144} = \sqrt{169} = 13 \neq 1.
\]
So, we should instead use the vector \((5/13, 12/13)\), b/c this is a unit vector parallel to \((5,12)\).

\[
\nabla F(\mathbf{e}_1, \mathbf{e}_2) = \left( \frac{\partial F}{\partial x}, \frac{\partial F}{\partial y} \right) \mathbf{e}_1 \cdot (\mathbf{e}_1, \mathbf{e}_2) = \left( y^x, y \ln x x^y \right) \mathbf{e}_1 \cdot (\mathbf{e}_1, \mathbf{e}_2) = \left( e^{-e^x}, e \ln x e^y \right) \mathbf{e}_1 \cdot (\mathbf{e}_1, \mathbf{e}_2)
\]
\[
= \left( e^{-e^x} \right) \left( \frac{5}{13}, \frac{12}{13} \right) = \frac{5}{13} e^{-e^x} + \frac{12}{13} e e = \frac{17}{13} e.
\]
5. Let \( F(x, y, z) = x^3 - y^3 + z^2 \). Find the maximum value for the directional derivative of \( F \) at the point \((1, 2, 3)\).

To find the maximum value for the directional derivative, we need to choose the unit vector \( \mathbf{v} \) which points in the direction along which \( F \) is increasing the fastest. By Theorem 13, we know this is \( \nabla F(1, 2, 3) \).

\[
\nabla F(1, 2, 3) = \left( \frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z} \right)_{(1,2,3)} = (3x^2, -3y^2, 2z) \big|_{(1,2,3)} = (3, -12, 6).
\]

The directional derivative of \( F \) at \((1, 2, 3)\) along \( \frac{(3, -12, 6)}{\sqrt{88}} \) is:

\[
\nabla F(1, 2, 3) \cdot \left( \frac{3, -12, 6}{\sqrt{88}} \right) = \frac{88 \cdot 2}{\sqrt{88}} = \sqrt{88} = 2\sqrt{21}.
\]

So, the maximum value for the directional derivative of \( F \) is \( \| \nabla F(1, 2, 3) \| \).
15. Show that the "target plane to level surfaces" definition yields, as a special case, the formula for the plane tangent to the graph of \( F(x,y) \) by regarding the graph as a level surface of \( F(x,y,z) = F(x,y) - z \).

Recall: For \( F: \mathbb{R}^2 \to \mathbb{R} \), the tangent plane of the graph of \( F \) at the point \((x_0, y_0, F(x_0, y_0))\) is defined by

\[ z = F(x_0, y_0) + \left[ \frac{\partial F}{\partial x}(x_0, y_0) \right] (x - x_0) + \left[ \frac{\partial F}{\partial y}(x_0, y_0) \right] (y - y_0). \] [Eq. 11b]

The graph of \( F(x,y) = z = F(x,y) \) is the level surface \( F(x,y,z) = 0 \). Using our "tangent plane to surfaces" definition, the tangent plane to this level surface is \( \nabla F(x_0, y_0, z_0) \cdot (x-x_0, y-y_0, z-z_0) = 0 \)

\[ \nabla F(x_0, y_0, z_0) = \left( \frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, -1 \right) \big|_{(x_0, y_0, z_0)} = \left( \frac{\partial F}{\partial x}(x_0, y_0), \frac{\partial F}{\partial y}(x_0, y_0), -1 \right). \]

So, \( \nabla F(x_0, y_0, z_0) \cdot (x-x_0, y-y_0, z-z_0) = 0 \)

\[
= \left[ \frac{\partial F}{\partial x}(x_0, y_0) \right] (x-x_0) + \left[ \frac{\partial F}{\partial y}(x_0, y_0) \right] (y-y_0) - z + z_0. \quad \text{But} \quad z = F(x,y)
\]

\(\Rightarrow z_0 = F(x_0, y_0). \) So \( \nabla F(x_0, y_0, z_0) \cdot (x-x_0, y-y_0, z-z_0) = 0 \)

becomes

\[
\left[ \frac{\partial F}{\partial x}(x_0, y_0) \right] (x-x_0) + \left[ \frac{\partial F}{\partial y}(x_0, y_0) \right] (y-y_0) + F(x_0, y_0) = z,
\]

So we can see that the \( z \) definition coincide.