Find the local maxima and minima for \( z = (x^2 + 3y^2)e^{1-x^2-y^2} \).

\[
\frac{\partial z}{\partial x} = 2x e^{1-x^2-y^2} + (x^2 + 3y^2) e^{1-x^2-y^2} \cdot (-2x) e^{1-x^2-y^2} = 2xe^{1-x^2-y^2} [1-x^2-3y^2] = 0 \iff x = 0 \text{ or } x = \pm \sqrt{3-y^2}.
\]

\[
\frac{\partial z}{\partial y} = 6ye^{1-x^2-y^2} + (x^2 + 3y^2) e^{1-x^2-y^2} \cdot (-2y) e^{1-x^2-y^2} = 6ye^{1-x^2-y^2} [3-x^2-3y^2] = 0 \iff y = 0 \text{ or } y = \pm \sqrt{3-x^2}.
\]

If \( x = 0 \) then \( y = 0 \) and \( x = \pm \sqrt{3-y^2} \), \( 3-y^2 = 0 \iff y^2 = 1 \iff y = \pm 1. \)

If \( y = 0 \) then \( x = 0 \) and \( x = \pm \sqrt{3-x^2} \), \( 1-x^2 = 0 \iff x^2 = 1 \iff x = \pm 1. \)

If \( x \neq 0 \) then \( 1-x^2-3y^2 = 0 \) and \( 3-x^2-3y^2 = 0 \).

So, our critical points are \((0,0), (0,1), (0,-1), (1,0), (-1,0)\).

\[
\frac{\partial^2 z}{\partial x^2} = \left[ 2e^{1-x^2-y^2} + 2x(-2x)e^{1-x^2-y^2} \right] [1-x^2-3y^2] + 2xe^{1-x^2-y^2} [-2x] = 2e^{1-x^2-y^2} [1-x^2-3y^2] + 4x^2e^{1-x^2-y^2} [-2x].
\]

\[
\frac{\partial^2 z}{\partial y^2} = \left[ 6e^{1-x^2-y^2} + (x^2 + 3y^2)e^{1-x^2-y^2} [3-x^2-3y^2] + 3ye^{1-x^2-y^2} [-2y] \right] = 6e^{1-x^2-y^2} [3-x^2-3y^2] + 3ye^{1-x^2-y^2} [-2y] = 6e^{1-x^2-y^2} [3-x^2-3y^2] + 3ye^{1-x^2-y^2} [-2y].
\]

\[
\frac{\partial^2 z}{\partial x \partial y} = \left[ 2e^{1-x^2-y^2} [1-x^2-3y^2] + 2xe^{1-x^2-y^2} [-2y] \right] = 2e^{1-x^2-y^2} [1-x^2-3y^2] + 2xe^{1-x^2-y^2} [-2y].
\]

\[
\left. \frac{\partial^2 z}{\partial x^2} \right|_{(0,0)} = 2e^0 \cdot 0 = 0, \left. \frac{\partial^2 z}{\partial y^2} \right|_{(0,0)} = 2e^0 \cdot 0 = 0, \left. \frac{\partial^2 z}{\partial x \partial y} \right|_{(0,0)} = 2e^0 \cdot 0 = 0.
\]

\[
D = \left. \left( \frac{\partial^2 z}{\partial x^2} \right) \right|_{(0,0)} \cdot \left. \left( \frac{\partial^2 z}{\partial y^2} \right) \right|_{(0,0)} - \left. \left( \frac{\partial^2 z}{\partial x \partial y} \right) \right|_{(0,0)}^2 = 12e^0 \cdot 0 = 0, \left. \left( \frac{\partial^2 z}{\partial x^2} \right) \right|_{(0,0)} \cdot \left. \left( \frac{\partial^2 z}{\partial y^2} \right) \right|_{(0,0)} = 12e^0 \cdot 0 = 0.
\]

\[
D = -12e^0 \cdot 0 = 0. \text{ Both } (0,0) \text{ and } (0,1) \text{ are local maxima.}
\]
(±1,0): \( \frac{\partial^2 f}{\partial x^2} |_{(±1,0)} = -4 \), \( \frac{\partial^2 f}{\partial y^2} |_{(±1,0)} = 2 \), \( \frac{\partial^2 f}{\partial x \partial y} |_{(±1,0)} = 4 \), \( \frac{\partial^2 f}{\partial x^2} |_{(±1,0)} = 0 \),

\[ D = 1 - 4(4) - 0^2 < 0. \] So, \((1,0) \neq (-1,0)\) are saddle points.

25. Let \( f(x,y) = x^2 - 2xy + y^2 \). Here \( D = 0 \). Can you say whether
the critical points are local minima, local maxima, or saddle points?

\[ D = 0 \Rightarrow \text{the second-derivative test doesn't tell us}
\]
anything, but we can still find the critical points by trying
to examine them using other methods:

\[ \frac{\partial f}{\partial x} = 2x - 2y, \quad \frac{\partial f}{\partial y} = -2x + 2y. \]

\[ \frac{\partial^2 f}{\partial x^2} = 2, \quad \frac{\partial^2 f}{\partial y^2} = 2, \quad \frac{\partial^2 f}{\partial x \partial y} = 0. \]

\[ \Rightarrow x = y. \quad \text{So all points of the form } (x,x) \text{ are critical}
\]
points (i.e., all points along the line } y = x \).

\[ f(x,x) = x^2 - 2x^2 + x^2 = 0. \]

Notice: \( f(x,y) = (x-y)^2 = (x-y)(x-y) = x^2 - 2xy + y^2 \geq 0 \),

\[ \Rightarrow \text{All points not on } y=x \text{ are local minima on the line } y=x.
\]

\[ \Rightarrow \text{All critical points } (x,x) \text{ are all local minima.} \]

27. Suppose \( f: \mathbb{R}^3 \to \mathbb{R} \) is \( C^2 \) and that \( x_0 \) is a critical point for \( f \). Suppose \( \nabla f(x_0)(h) = \mathbf{h}_1 \mathbf{h}_1 + \mathbf{h}_2 \mathbf{h}_2 + \mathbf{h}_3 \mathbf{h}_3 \).

Does \( f \) have a local max, min, or saddle at \( x_0 \)?

Recall: \( \nabla f(x_0)(h) \) is positive definite if \( f(x_0)(h) = 0 \) \iff \( h = 0 \).

\[ \nabla f(x_0)(h) \] is negative definite if \( f(x_0)(h) = 0 \) \iff \( h = 0 \).

Hence \( \nabla f(x_0)(h) = \frac{\partial}{\partial x_1} \mathbf{h}_1 \mathbf{h}_1 + \frac{\partial}{\partial x_2} \mathbf{h}_2 \mathbf{h}_2 + \frac{\partial}{\partial x_3} \mathbf{h}_3 \mathbf{h}_3 \).

\[ \nabla f(x_0)(h) = \mathbf{h}_1 \mathbf{h}_1 + \mathbf{h}_2 \mathbf{h}_2 + \mathbf{h}_3 \mathbf{h}_3 \]

\( \Rightarrow \) \( f \) at \( x_0 \).
Theorem 5 [2nd Derivative Test for Local Extrema]: \( \text{If } f: \mathbb{R}^n \to \mathbb{R} \) is \( C^3 \), \( x_0 \in U \) a critical point of \( f \), and \( \nabla^2 f(x_0) \) is positive definite, then \( x_0 \) is a relative minimum.

If \( \nabla^2 f(x_0) \) is positive definite and \( \nabla^2 f(x_0) \) is not negative definite, then \( x_0 \) is a saddle point.

Here \( \nabla^2 f(x_0) \) is not positive definite and not negative definite. Indeed,

\[
\nabla^2 f(x_0) (\mathbf{u}, -1, 1) = (5\mathbf{u})^2 + (-1)^2 + 2 + 4(-1) = 2 + 1 + 1 - 4 = 0.
\]

But \( h = (\mathbf{u}, -1, 1) \neq 0 \).

\[ \therefore x_0 \text{ is a saddle point.} \]