Math 2203: Week #9 Practice Problems

3.5: # 1, 3, 7, 9, 13, 17, 18.

1. Show that the eqn \( x + y - z + \cos(xyz) = 0 \) can be solved for \( z = g(x, y) \) near the origin. Find \( \frac{\partial g}{\partial x} \) and \( \frac{\partial g}{\partial y} \) at \((0,0)\).

Recall: Theorem 11: Special Implicit Function Theorem:

Suppose \( F: \mathbb{R}^{n+1} \rightarrow \mathbb{R} \) has continuous partial derivatives.

Assume \((x_0, z_0)\) satisfies \( F(x_0, z_0) = 0 \) and \( \frac{\partial F}{\partial z}(x_0, z_0) \neq 0 \).

Then \( \exists \) a ball \( U \) containing \( x_0 \) and a nbhd \( V \) of \( z_0 \) s.t. \( \exists \) a function \( z = g(x) \) defined for \( x \in U \) and \( z \in V \) s.t. \( F(x, g(x)) = 0 \). If \( x \in U \) and \( z \in V \) satisfy \( F(x, z) = 0 \), then \( z = g(x) \) is \( C^1 \) with derivative given by:

\[
Dg(x) = \left. \frac{\partial F}{\partial z}(x, z) \right|_{z=g(x)}
\]

Here we have \( F: \mathbb{R}^3 \rightarrow \mathbb{R} \)

\((x, y, z) \mapsto x + y - z + \cos(xyz) \).

\((x_0, z_0) = (0,0, z_0)\).

\( F(x, z) = \begin{bmatrix} \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} & \frac{\partial F}{\partial z} \end{bmatrix} = \begin{bmatrix} 1 - \sin(xyz)yz & 1 - \sin(xyz)x \end{bmatrix} \)

So, we can see \( F \) has continuous partials. Also,

\( F(x, y, z) = 0 \) \( \forall (x, y, z) \in \mathbb{R}^3 \). So, in particular,

\( F(x_0, z_0) = F(0,0, z_0) = 0 \).

\( \frac{\partial F}{\partial z}(x_0, z_0) = \frac{\partial F}{\partial z}(0,0, z_0) = -1 - \sin(0) \cdot 0 = -1 \neq 0 \).

\( \therefore \) By Theorem 11 \( \exists \) a ball \( U \subseteq \mathbb{R}^3 \) containing \((0,0, z_0)\)
a nbhd $V \subseteq \mathbb{R}^n$ containing $z_0$ s.t. $F$'s function

$z = g(x)$ for $x \in U \ni z \in V$ s.t. $F(x, g(x)) = 0$.

$\therefore$ The eqn $x+y - z + \cos(xyz) = 0$ can be solved for $z = g(x,y)$ near $(0,0)$.

Also, by Theorem 11 we have:

$$Dg(x) = \left[ \frac{\partial g}{\partial x} \frac{\partial g}{\partial y} \right] = -\frac{1}{\frac{\partial F(x,z)}{\partial z}} \begin{bmatrix} \frac{\partial F}{\partial x} \\ \frac{\partial F}{\partial y} \end{bmatrix}, \quad z = g(x)$$

$\Rightarrow Dg(0) = \left[ \frac{\partial g}{\partial x} \frac{\partial g}{\partial y} \right]_{(0,0)} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. 

\[ \text{Check: } \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \text{and } \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \]
7. Show that \( x^3 y^2 - z^3 y x = 0 \) is solvable for \( z \) as a function of \((x, y)\) near \((1, 1, 1)\), but not near the origin. Compute \( \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} \).

\[
F(x, y, z) = x^3 y^2 - z^3 y x.
\]

\[
DF = \begin{bmatrix}
3x^2 y^2 - z^3 y & -z^3 x
\end{bmatrix}
\]

So, \( F \) has continuous partials. Here \( x_0 = (1, 1) \) and \( z_0 = 1 \).

\[
\frac{\partial F}{\partial z}(1, 1, 1) = 2 - 3 = -1 \neq 0.
\]

So, we can write \( z = g(x, y) \) for some function \( g \) near \((1, 1, 1)\). However, \( \frac{\partial F}{\partial z}(0, 0, 0) = 0 \) so we can't use Theorem 11 to write \( z \) as a function of \((x, y)\) near \((0, 0, 0)\). In particular, in a nbhd of \((0, 0)\) we have \( x \neq 0 \) and \( y \neq 0 \). Suppose \( F(x, y, g(x, y)) \) where \( y = x \) we have

\[
\frac{\partial z}{\partial x} = 1 \quad \text{and} \quad F(x, y, g(x, y)) = 0.
\]

\[
z^2 \left[ x^3 - z y x \right] = 0 \Rightarrow z = 0 \quad \text{or} \quad x^3 - z y x = 0.
\]

In the nbhd where \( y = x \) we have that this eqn would be satisfied for \( z = 0 \) or \( z = x \). So, since
Here is no unique value of $z$. You can be no function in this nbhd. s.t. $g(x, y) = z$.

$\forall x \in g(x, x) = 0$ & $g(x, x) = x$ works.

$Dg(1, 1) = \left[ \frac{\partial g}{\partial x}, \frac{\partial g}{\partial y} \right] = \left[ \frac{-1}{\frac{\partial F}{\partial x}(1, 1, 1)}, \frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}(1, 1, 1)} \right] = \frac{1}{-1} \left[ 2, 1 \right] = \left[ a, b \right]$. 
17. Consider the equations:
\[ x^2 - y^2 - u^3 + v^2 + 4 = 0 \]
\[ 2xy + y^2 - 2u^2 + 3v^4 + 8 = 0. \]

(a) Show that these equations determine functions \( u(x, y, v) \) and \( v(x, y) \) near the point \( (x, y, u, v) = (2, 1, 2, 1) \).

\[
\Delta = \begin{vmatrix}
\frac{\partial F}{\partial u} & \frac{\partial F}{\partial v} \\
\frac{\partial F}{\partial u} & \frac{\partial F}{\partial v}
\end{vmatrix}
= \begin{vmatrix}
-3u^2 & 2v \\
-4u & 12v^3
\end{vmatrix}
= (2, 12, 1)
\]

\[
= \begin{vmatrix}
-12 & 2 \\
-8 & 12
\end{vmatrix}
= -144 + 16 \neq 0.
\]

So, by Theorem 12, these equations determine functions \( u(x, y) \) and \( v(x, y) \) near \( (x, y, u, v) = (2, 1, 2, 1) \).

(b) Compute \( \frac{\partial u}{\partial x} \) at \( (x, y) = (2, 1) \).

Using implicit differentiation, we have:
\[ 2x - 3u^2 u' + 2uv' = 0 \quad \text{and} \quad 2y - 4uu' + 12v^3 v' = 0. \]

At \( (2, 1) \), we have:
\[ u' = 0 \quad \text{and} \quad v' = 0. \]
\[
\begin{align*}
4 - 12u' + 2v' &= 0 \\
-24 + 72u' - 12v' &= 0 \\
-2 - 8u' + 12v' &= 0 \\
-26 + 64u' &= 0
\end{align*}
\]

\[\frac{v'}{u'} = \frac{26}{64} = \frac{13}{32}.\]

10. It is possible to solve the system of equations.