Math 2XX3: Week #3 Practice Problems

6.4: #1, 3, 5, 7, 17, 18
4.3: #1, 3, 5, 9, 10, 11, 13, 14, 17, 19, 21, 27

6.4: 1. Evaluate $\iint_D \frac{1}{xy} \, dA$, where $D = [0,1] \times [0,1]$ if it exists.

Recall: Theorem 3 [Fubini's Theorem]: Let $D$ be an elementary region in the plane at $F \geq 0$

a cont. function except for possibly points on the boundary of $D$. If either of the integrals $\iint_D F(x,y) \, dA$,

\[ \int_0^1 \int_0^1 F(x,y) \, dy \, dx \quad \text{for } y\text{-simple regions} \]

\[ \int_0^1 \int_0^1 F(x,y) \, dx \, dy \quad \text{for } x\text{-simple regions exist as improper integrals,.then } F \text{ is integrable if they are all equal}. \]

\[ F = \frac{-1}{\sqrt{xy}} \geq 0 \text{ cont. except at the points } (x,0), (y,0) \text{ on the boundary of } D. \]

It is both $x$- and $y$-simple, but here let's view it as $y$-simple ($x^2 + y^2 = 1$, $y = 0 \text{ at } x = 1$, $y = 1 \text{ at } x = 0$).

\[ \phi_1(x) = 0 \quad \phi_2(x) = 1 \quad \phi_1(x) = 0 \]

\[ \int_0^1 \int_0^1 F(x,y) \, dy \, dx = \int_0^1 \int_0^1 (xy) \, dy \, dx = \int_0^1 \frac{x}{2} \, dx = \frac{x^2}{4} \bigg|_0^1 = \frac{1}{4} \]

\[ \int_0^1 \int_0^1 F(x,y) \, dx \, dy = \int_0^1 \int_0^1 (xy) \, dy \, dx = \int_0^1 \frac{y}{2} \, dy = \frac{y^2}{4} \bigg|_0^1 = \frac{1}{4} \]

So, by Fubini's Theorem, $\iint_D \frac{1}{xy} \, dA = \frac{1}{4}$. \]
3. \( \iint_D \frac{y}{x} \, dx \, dy \), where \( D \) is bounded by \( x = 1 \), \( x = y \), \( y = x^2 \), \( y = 2x \).

Let's view this as the \( y \)-simple region.

In \( D \) we have \( x > 0 \) and \( y > 0 \) \( \Rightarrow F + \frac{y}{x} > 0 \) on \( D \). \( \frac{y}{x} \) cont. except when \( x = 0 \) (i.e., \( x = y \)).

\[
\begin{align*}
\int_0^1 \int_{1/2}^1 \frac{y}{x} \, dy \, dx &= \int_0^1 \left[ \frac{1}{2} y^2 \right]_{1/2}^1 \, dx \\
&= \int_0^1 \frac{1}{2} \left( \frac{1}{2} - 1 \right) \, dx \\
&= \frac{1}{2} \int_0^1 -\frac{1}{4} \, dx \\
&= \frac{1}{2} \left[ -\frac{1}{4} x \right]_0^1 \\
&= \frac{1}{2} \left( -\frac{1}{4} \right) \\
&= \frac{3}{16}.
\end{align*}
\]

So, by Fubini's Theorem, \( \iint_D \frac{y}{x} \, dx \, dy = \frac{3}{16} \).
7. (a) Evaluate \( \iint_D \frac{dA}{(x^2+y^2)^{3/2}} \), where \( D \) is the unit disk in \( \mathbb{R}^2 \).

We can see that our function \( f(x,y) = \frac{1}{(x^2+y^2)^{3/2}} \) is defined everywhere, except at the point \((0,0)\). Also, \( f(x,y) \neq 0 \).

Recall (Functions Unbounded at Isolated Points, pg. 344): Let \((x_0, y_0)\) be a point of a general region \( D \) where a function \( F \) is undefined.

If \( F \) is continuous at every point of \( D \), except \((x_0, y_0)\), then \( \iint_D F \, dA \) is defined, where \( D_\varepsilon = D \setminus B(x_0, y_0, \varepsilon) \) disk of radius \( \varepsilon \) centered at \((x_0, y_0)\). \( \iint_D F \, dA \) is convergent (or \( F \) is integrable) over \( D \) if \( \lim_{\varepsilon \to 0} \iint_{D_\varepsilon} F \, dA \) exists.

\[
\iint_D \frac{dA}{(x^2+y^2)^{3/2}} = \int_0^\infty \int_0^{\pi/2} \frac{r}{r^3} \, dr \, d\theta = 2\pi \int_0^1 \frac{1}{r^2} \, dr = 2\pi \left[ -\frac{1}{r} \right]_1^1 = -2\pi \left( 0 - 1 \right) = 2\pi.
\]

(b) Determine the real numbers \( \lambda \) for which the integral \( \iint_D \frac{dA}{(x^2+y^2)^2} \) is convergent, where \( D \) is the unit disk.

If \( \lambda \leq 0 \), then \( \frac{1}{(x^2+y^2)^{\lambda}} \) is continuous and defined everywhere \( \Rightarrow \iint_D \frac{dA}{(x^2+y^2)^{\lambda}} \) converges over \( D \).

If \( \lambda > 0 \) we have:

\[
\iint_D \frac{dA}{(x^2+y^2)^2} = \int_0^\pi \int_0^1 \frac{1}{r^2} \, dr \, d\theta = 2\pi \int_0^1 \frac{1}{r} \, dr = 2\pi \left[ -\ln r \right]_1^1 = 2\pi \left( 0 - 0 \right) = 0.
\]

So, \( \lambda = 0 \).

\[
\iint_D \frac{dA}{(x^2+y^2)^2} = \frac{\pi}{1-\lambda} \text{ if } 2-2\lambda \geq 0 \text{ and } \infty \text{ otherwise. So we need } 2-2\lambda \leq 1.
\]
If \( \lambda = 1 \) we have:

\[
2\pi \int_0^1 F^{-1} \, dr = 2\pi \ln(1/\delta)
\]

\[
= 2\pi [\ln(1) - \ln(\delta)] = -2\pi \ln(\delta).
\]

\[
\lim_{\delta \to 0} 2\pi \ln(\delta) = \infty.
\]

\[
\therefore \iiint_D \frac{da}{(x+y+z)^2} \text{ is convergent over } D \text{ if } \lambda < 1.
\]
18. Let $D$ be the unbounded region defined as $\{(x,y,z) \mid x^2 + y^2 + z^2 \leq 1\}$.

By making a change of variables, evaluate the improper integral

$$\int\int\int_D \frac{1}{(x^2+y^2+z^2)^2} \, dx \, dy \, dz.$$
\[ \lim_{\theta \to 0} \int_0^\infty \frac{e^{-r} \cos \theta}{r} \, dr = 0 \]

\[ \lim_{r \to 0} \int_0^\infty \frac{e^{-r} \cos \theta}{r} \, dr = 1 \]

\[ = -4 \pi \left[ 0 - 1 \right] = 4 \pi. \]

**4.3**

1. Sketch the vector field \( \mathbf{F}(x,y) = (2x, 2y) \) (or a small multiple of it).

At every \( (x,y) \) have a vector with the same positive slope and with the same length.

2. Sketch \( \mathbf{F}(x,y) = (x, y) \).

3. Sketch \( \mathbf{F}(x,y) = (x, y) \).

5. Sketch \( \mathbf{F}(x,y) = (2y, x) \).

At each point \( (x,y) \) sketch \( (2y, x) \). Magnitude increases as move away from origin.

\[(1, 1) \rightarrow (2, 2), (2, 1) \rightarrow (4, 1), (0, 1) \rightarrow (0, 2), (1, 0) \rightarrow (2, 0), (0, 0) \rightarrow (0, 0).\]
11. \( \text{Sketch a few flow lines for the vector fields.} \)

11. \( F(x, y) = (2y, x-y) \).

Recall: \( F \) is a vector field, a flow line for \( F \) is a path \( c(t) \) s.t. \( c'(t) = F(c(t)) \).

11. \( F(x, y) = (y, -x) \).

The vector field looks like:

The vector field looks like circles:

13. \( F(x, y) = (x, x^2) \).

14. \( F(x, y, z) = (y-z, x, x^2) \).

[See 4.3.10 pg. 241]
Looks like parabolas $y = x^2 + c$. 

Let $c(t) = (\pm \sqrt{t^2 + c})$. 

$c(t) = (1, \pm t)$. 

$F(c(t)) = (t, \pm y)$. 

Not a good parametrization.

Try: 

$c(t) = (e^{t}, e^{t^2} + c)$. 

c'(t) = (e^{t}, 2te^{t^2}).$ 

$F(c(t)) = (e^{t}, e^{t^2})$. 

$c(t) = (-e^{t}, \frac{1}{2} e^{t^2} + c)$. 

c'(t) = (-e^{t}, e^{t^2}).$ 

$F(c(t)) = (-e^{t}, e^{t^2})$. 

Score!