Find the length of the path \( c(t) \), defined by:
\[
\begin{align*}
c(t) &= (2 \cos t, 2 \sin t, t) & \text{if} & \quad 0 \leq t \leq 2 \pi \\
c(t) &= (2, 1, t) & \text{if} & \quad 2 \pi \leq t \leq 4 \pi.
\end{align*}
\]

Recall: The arc length of a path \( c(t) = (x(t), y(t), z(t)) \)

For \( t_0 \leq t \leq t_1 \) is:

\[
L(c) = \int_{t_0}^{t_1} \| c'(t) \| \, dt.
\]

We take the length of each piece, then add them together:

\[
c'(t) = (-2 \sin t, 2 \cos t, 1) & \text{ if } 0 \leq t \leq 2 \pi. \\
\int_0^{2\pi} \sqrt{4 \sin^2 t + 4 \cos^2 t + 1} \, dt = \int_0^{2\pi} \sqrt{5} \, dt = 2 \pi \sqrt{5}.
\]

\[
c'(t) = (0, 1, 1) & \text{ if } 2 \pi \leq t \leq 4 \pi. \\
\int_{2\pi}^{4\pi} \sqrt{2} \, dt = \sqrt{2} [4 \pi - 2 \pi] = 2 \pi \sqrt{2}.
\]

\[
\therefore \text{The length of } c(t) \text{ is } 2 \pi \left( \sqrt{5} + \sqrt{2} \right)
\]
8. The intersection of the cylinder $y^2 + z^2 = 1$ and the plane $z = x$.

$c(t) = (\sin t, \cos t, \sin t)$ for $t \in [0, 2\pi]$.

$c(t) = (\sin(2\pi t), \cos(2\pi t), \sin(2\pi t))$ for $t \in [0, 1]$.

II. Evaluate the following path integrals $\int_c F(x,y,z) \, ds$, where

(a) $F(x,y,z) = e^{x^2}$, and $c: t \mapsto (1, 2t^2, t)$, $t \in [0, 1]$.

Recall: The path integral of $F(x,y,z)$ along the path $c : I \to \mathbb{R}^3$ is defined when $c : [a,b] \to \mathbb{R}^3$ is cont. on $I$. Then

$$\int_c F \, ds = \int_a^b \| F(c(t)) \| \, ||c'(t)|| \, dt.$$ 

$F(0t) = e^t = e^t$.

$c'(t) = (0, 0, 1)$.

$||c'(t)|| = \sqrt{1t^2} = at$.

$$\int_0^1 e^t \, dt = e - e^0 = 0.$$ 

(a) $F(x,y,z) = yz$ and $c: t \mapsto (t, 3t, 2t)$, $t \in [1,3]$.

$$\int_c F \, ds = \int_1^3 (3t)(2t) ||1 + t + 4| \, dt$$

$$= \int_1^3 6t^2 \, dt = \frac{1}{3} \int_1^{14} t^3 \, dt = \frac{1}{3} \int_1^{14} t^3 \, dt = \frac{1}{3} \int_1^{14} t^3 \, dt = \frac{1}{3} \left[ \frac{1}{4} t^4 \right]_1^{14} = \frac{1}{3} \left[ \frac{1}{4} (14^4) - \frac{1}{4} (1^4) \right] = \frac{1}{3} \left[ 14^4 - 1 \right] = \frac{1}{3} \left( 14 - 1 \right) = \frac{1}{3} \left( 21 - 1 \right) = \frac{1}{3} \left( 20 \right) = \frac{20}{3}.$$

$$= 52 \sqrt{14}.$$
Reparametrize \( c(t) = (a \cos t, a \sin t) \) by arc length and compute the curvature wise at \( 2\pi \).

**Definition:** A path \( c(s) \) is parametrized by arc length if \( ||c'(s)|| = 1 \).

To do this, we'll want to reparametrize using the arc length function \( s(t) = \int_0^t ||c'(t)|| \, dt \).

Hence \( s(t) = \int_0^t ||c'(t)|| \, dt \)

\[
= \int_0^t a \, dt = a \, t \Rightarrow t = \frac{s}{a} \quad s(t) = \frac{a}{a} s \quad t = 0 \Rightarrow s = 0, \quad t = 2\pi \Rightarrow s = 2\pi a.
\]

So, \( c(s) = (a \cos (\frac{a}{a}s), a \sin (\frac{a}{a}s)) \), \( s \in [0, 2\pi a] \)

\( c(s) \) still cuts our circle and it should have \( ||c'(s)|| = 1 \).

\[
||c'(s)|| = \left( \frac{a \sin (\frac{a}{a}s)}{a}, -\frac{a \cos (\frac{a}{a}s)}{a} \right) = 1. \checkmark
\]

**Recall:** The curvature of a path parametrized by arc length is \( k = ||c''(s)|| \).

So, the curvature of our curve is:

\[
k = ||c''(s)|| = \left( \frac{a \sin (\frac{a}{a}s)}{a^2}, -\frac{a \cos (\frac{a}{a}s)}{a^2} \right) = \frac{1}{a^2}.
\]

[Circle curves equally at every point.]

[Curvature constant... circles of small radius have large curvature.

[Curvature measures the intensity of the force needed to keep the particle on the track of \( c(s) \).]

[Measures how quickly the curve changes direction... measures the rate of change of direction vector as \( s \) changes.]

[curvature of a line is zero.]
Consider the force field \( F(x, y, z) = (x, y, z) \). Compute the work done in moving a particle along the parabola \( y = x^2, z = 0 \), from \( x = -1 \) to \( x = 2 \).

\[
\mathbf{c}(t) = (t, t^2, 0) \text{ for } t \in [-1, 2], \quad \mathbf{c}'(t) = (1, 2t, 0).
\]

\[
\oint_{C} \mathbf{F} \cdot d\mathbf{s} = \int_{-1}^{2} (t, t^2, 0) \cdot (1, 2t, 0) \, dt = \int_{-1}^{2} t + 2t^3 \, dt
\]

\[
= \left. \frac{1}{2}t^2 + \frac{1}{4}t^4 \right|_{-1}^{2} = 4 + 8 - \left[ \frac{1}{2} + \frac{1}{4} \right] = 9.
\]
17. Evaluate the line integral \( \int_C 2xyz \, dx + x^2z \, dy + y^2z \, dz \), where \( C \) is an oriented simple curve connecting (1,1,1) to (1,2,4).

Since \( F = \nabla (xyz) = (2xyz, x^2z, y^2z) \), it suffices to find any parametrization of a simple curve from (1,1,1) to (1,2,4).

Recall: A map \( c: I \rightarrow \mathbb{R}^3 \) is called a parametrization of a simple curve \( C \) if it is \( C \) injective on \( I \).

So, it suffices to find a \( C' \) injective map from (1,1,1) to (1,2,4).

\( C(t) = (1, t, 3t^2 - 2) \) for \( t \in [1,2] \).

\( C'(t) = (0, 1, 3) \).

\[
\int_{C} 2t(3t^2 - 2)(0) + (3t^2 - 2)(1) + t(3) \, dt = \int_{1}^{2} 3t^2 - 2 + 3t \, dt = \int_{1}^{2} 3t^2 + 3t - 2 \, dt = \left[ t^3 + \frac{3}{2}t^2 - 2t \right]_{1}^{2} = 2^3 + \frac{3}{2} \cdot 2^2 - 2 \cdot 2 - (1^3 + \frac{3}{2} \cdot 1^2 - 2 \cdot 1) = 7.
\]

Note: You could have also solved this question by finding that \( F = (2xyz, x^2z, y^2z) \) is equal to \( \nabla F \), where \( F(x,y,z) = x^2yz + C \). Then, by Theorem 3, \( \int_C F \cdot ds = \int_C \nabla F \cdot ds = F(1,2,4) - F(1,1,1) = (8 + C) - (1 + C) = 7. \)

\[
\frac{\partial F}{\partial x} = 2xyz \quad \frac{\partial F}{\partial y} = x^2z \quad \frac{\partial F}{\partial z} = y^2z.
\]

So, \( F = \sqrt{x^2yz + C} \).