Math 2203 - Tutorial #1

ODE vs. PDE: ODE has a single independent variable, whereas a PDE has at least two independent variables.

\[ \frac{d^2 y}{dx^2} + \frac{dy}{dx} + 8 = 0 \] [O.D.E.]

\[ \frac{\partial^2 y}{\partial x^2} + \frac{\partial y}{\partial x} + 8 = 0 \] [P.D.E.]

- **Order**: highest derivative.

4. State the order of the following ODE. Linear?

- (a) \[ \frac{d^2 y}{dx^2} = \sqrt{1 + (\frac{dy}{dx})^2} \]
  - Order: 2. Not linear b/c \( \frac{dy}{dx} \) under square root.

- (b) \( 3\sin x \) \( y'' - 3\cos x y' = 10 \). Order 3. Linear.

5. Is the piecewise defined function a solution to the given DE on the interval provided? Explain.

- (a) \( y = \begin{cases} 5 - x^2, & x < 0 \\ x^2, & x \geq 0 \end{cases} \)
  - \( y \) is a solution on \(-\infty, 0) \) and \((0, \infty) \) since it satisfies the differential equation on those intervals.
3. Given that \( y = \sin x \) is an explicit solution of \( y' = \sqrt{1 - y^2} \). Find an interval of definition.

We need to find an interval \( I \) such that \( y = \sin x \) is \( C^1 \) on \( I \) and

when \( y = \sin x \) is plugged into the eqn \( y' = \sqrt{1 - y^2} \), it reduces to the identity.

\[ y = \sin x \] is \( C^1 \) everywhere, since \( \frac{dy}{dx} = \cos x \) is continuous everywhere.

Plugging \( y = \sin x \) into the eqn:

\[ y' = \sqrt{1 - y^2} \]

\[ \Rightarrow \cos x = \sqrt{1 - \sin^2 x} \]

\[ \Rightarrow \cos x = \sqrt{\cos^2 x} \]

\[ \Rightarrow \cos x = |\cos x| \]. This is only true for those values of \( x \) where \( \cos x \) is positive. So we can choose any interval where \( \cos x \) is positive. In particular, \( I = (-\frac{\pi}{2}, \frac{\pi}{2}) \) does the job.
2.a) Verify that the piece-wise-defined function
\[ y = \begin{cases} 
  x^2 & \text{if } x < 0 \\
  x^2 & \text{if } x \geq 0
\end{cases} \]
is a solution of the DE \( xy' - 2y = 0 \) on \( (-\infty, \infty) \).

The DE is 1st-order, so must show \( y \) satisfies
DE \( \phi \) is \( C^1 \) on \( (-\infty, \infty) \).

\[ x < 0 : \quad \frac{dy}{dx} (x^2) = -2x, \quad xy' - 2y = x(-2x) - 2(-x^2) = -2x^2 + 2x^2 = 0. \]
\[ x \geq 0 : \quad \frac{dy}{dx} (x^2) = 2x, \quad xy' - 2y = x(2x) - 2(x^2) = 0. \]

Need to check if \( y' \) continuous on \( (-\infty, \infty) \).

\[ y' = \begin{cases} 
  -2x^2 & \text{if } x < 0 \\
  2x & \text{if } x \geq 0
\end{cases} \]
\[ \lim_{x \to 0^-} y' = \lim_{x \to 0^-} -2x = 0. \]
\[ \lim_{x \to 0^+} y' = \lim_{x \to 0^+} 2x = 0. \]

This implies DE \( C^1 \) on \( (-\infty, \infty) \) \( \implies \)
\( y \) is a solution on \( (-\infty, \infty) \).

b) In Ex. 5 we saw \( y = \sqrt{a} \sqrt{5 - x^2} \) and \( y = -\sqrt{a} \sqrt{5 - x^2} \)
are solutions of \( y' = -\frac{x}{y} \) on \( (-5, 5) \). Explain why
\[ y = \begin{cases} 
  \sqrt{\frac{25}{x^2}} & \text{if } -5 < x < 0 \\
  -\sqrt{\frac{25}{x^2}} & \text{if } 0 \leq x < 5 \not \text{ a solution to DE on } (-5, 5).
\end{cases} \]

We can see \( y \) is not
\[ \lim_{x \to 0^-} y = 5. \]
\[ \lim_{x \to 0^+} y = -5. \]
4. Determine whether the Existence/Uniqueness Theorem guarantees that the DE \( y' = \sqrt{y^2 - 9} \) possesses a unique solution through the given point.

- (a) \((1, 4)\)
- (b) \((2, -3)\)

Here \( f(x, y) = \sqrt{y^2 - 9} \). Note: The Theorem is satisfied for points considered on the interior of a region \( R \), where \( f \) and \( \frac{df}{dy} \) are continuous on \( R \).

\[
f(x, y) = \sqrt{y^2 - 9}
\]

This is continuous as long as \( y^2 - 9 \geq 0 \).

\[
y^2 - 9 \geq 0 \Rightarrow y^2 \geq 9 \Rightarrow |y| \geq 3 \Rightarrow y \geq 3 \text{ or } y \leq -3.
\]

So, we could choose any points \((x, y)\) satisfying the interval of \((-\infty, -3] \cup [3, \infty)\).

\[(x+\delta, y+\delta) \Rightarrow (3, 0) \quad \text{is satisfied for } (1, 4) \text{.}
\]

- (b) \(-3 \not\in (-\infty, -3) \cup (3, \infty)\), so Theorem Not satisfied for \((2, -3)\).
5. Suppose you're given a 1st-order DE $y' = f(x, y)$, where $f$ & $\partial f/\partial y$ are cont. on a rectangular region $R$. Could a solution curve in its 1-param. family of solutions intersect at a point in $R$? Why or why not?

- If $f$ & $\partial f/\partial y$ cont. on $R$ then for any point $(x_0, y_0)$ in $R$, a solution exists & is unique in some neighborhood around that point (by Existence/Uniqueness Theorem). Therefore, in that nbhd. 2 solutions can't intersect because there's a unique solution in that nbhd.
  - This is true for all pts in $R$.
  - $\therefore$ 2 curves can't intersect in $R$.

6. Match the DE to the direction field. [See slides 7.]

- $a) y' = (2-x) (3+x)$
- $b) y' = (y-a) (3-y)$
- $c) y' = (a-x) (3+y)$

Looking at 1st Direction Field: $y' = 0$ at $y = 2$  $\Rightarrow y' = 3$  $\Rightarrow$ could be $b$ or $a$.

When $2 < y < 3$, $y' > 0$ $\Rightarrow$ it must be $b$.  $\square$
3rd Direction Field: Also \( y' = 0 \) at \( y = 2 \) \& \( y = 3 \) and for \( 2 < y < 3 \) we have \( y' < 0 \) \( \Rightarrow \) cor. to \( \Box \). \( \Box \leftrightarrow \Box \).

2nd Direction Field: \( y' = 0 \) at \( x = 2 \) \& \( x = -3 \) \( \Rightarrow \) cor. to \( \Box \). \( \Box \leftrightarrow \Box \).

[\Box \) doesn't cor. to anything\(].