Find the minimum radius of convergence of a power series solution of \((x^2-3)y'' + 2xy' + y = 0\) about the ordinary point: \(x=0\), \(x=1\).

Recall: A point \(x_0\) is said to be an ordinary point of the DE if both \(P(x)\) and \(Q(x)\) in standard form \(y'' + P(x)y' + Q(x)y = 0\) are analytic at \(x_0\). A point that is not an ordinary point is called a singular point.

Recall: A function \(F\) is analytic at a point \(x_0\) if it can be represented by a power series in \(x-x_0\) with a positive radius of convergence. It can be written in the form \(\sum_{n=0}^{\infty} c_n (x-x_0)^n\) on an interval \(R\) s.t. for all \(y \in R\) we have that \(\lim_{n \to \infty} \frac{c_n}{n!} (x-x_0)^n\) exists.

Fact: If \(a_0\) and \(a_1\) are polynomials with no common factors, then \(a_0y'' + a_1y' + a_0y = 0\) has \(x=x_0\) as an ordinary pt iff \(a_0x_0 \neq 0\).

Recall: Existence of Power Series Solution: If \(x=x_0\) is an ordinary pt of \(a_0y'' + a_1y' + a_0y = 0\), then we can always find a l.i.n. ind. Power Series solutions centered at \(x_0\) with minimum radius of convergence \(R\), where \(R\) is the distance to the closest singular point.

\[ y = \sum_{n=0}^{\infty} c_n (x-x_0)^n. \]

1. \(x=0\): Only singular pts are when \(x^2-150 \equiv x=\pm 5\).

\[ |x-0| = 5 \equiv R=5. \]

2. \(x=1\):

\[ |x-1| = 4 \equiv R=4. \]
b) Find the minimum radius of convergence of a power series solution of 
\((x^2 - 2x + 10)y'' + xy' - 4y = 0\) about the ord. pt. \(x = 1\).

\[x^2 - 2x + 10 = 0 \iff x = 2 \pm \frac{\sqrt{4 - 40}}{2} = 1 \pm 3i.\]

\[R = \sqrt{(1-1)^2 + (0-3)^2} = 3 \iff R = 3.\]

c. a) Find the first 5 nonzero terms in the general solution to \((1+x^2)y'' - y' + y = 0\).

b) Explain why this is the general solution.

i.e.: Identify 2 linearly independent solutions.

\[y = \sum_{n=0}^{\infty} c_n x^n, \quad y' = \sum_{n=1}^{\infty} n c_n x^{n-1}, \quad y'' = \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}.\]

\[0 = (1+x^2)y'' - y' + y = (1+x^2)\sum_{n=2}^{\infty} c_n n(n-1) x^{n-2} - \sum_{n=1}^{\infty} n c_n x^{n-1} + \sum_{n=0}^{\infty} c_n x^n.\]

\[= \sum_{n=2}^{\infty} c_n n(n-1) x^{n-2} + \sum_{n=1}^{\infty} n c_n x^{n-1} - \sum_{n=0}^{\infty} c_n x^n.\]

\[= \sum_{n=2}^{\infty} c_n n(n-1) x^{n-2} + \sum_{n=1}^{\infty} n c_n x^{n-1} - \sum_{n=0}^{\infty} c_n x^n.\]

\[= \sum_{k=0}^{\infty} c_{k+2} (k+2)(k+1) x^k + \sum_{k=2}^{\infty} c_k k(k-1) x^k - \sum_{k=0}^{\infty} c_{k+1} (k+1) x^k + \sum_{k=0}^{\infty} c_k x^k.\]

\[= (2c_2 + 6c_3 x) - (c_1 + 2c_2 x) + (c_0 + c_1 x)\]

\[+ \sum_{k=3}^{\infty} (c_{k+2} (k+2)(k+1) + c_k [k(k-1)+1] - c_{k+1} (k+1)) x^k.\]
The first terms in the general solution are:

\[ y = \sum_{n=0}^{\infty} c_n x^n \]

\[ = c_0 + c_1 x + \frac{(c_1 - c_0)}{2} x^2 + \left(-\frac{1}{6} c_0\right) x^3 + \frac{-3 c_1 + 3 c_0}{24} x^4 + \ldots \]

Choosing \( c_0 = 0 \) and \( c_1 = 1 \) we have:

\[ y_1 = x + \frac{1}{2} x^2 - \frac{3}{24} x^4 + \ldots \]

Choosing \( c_0 = 1 \) and \( c_1 = 0 \) we have:

\[ y_0 = 1 - \frac{1}{2} x^2 - \frac{1}{6} x^3 + \frac{3 a}{64} x^4 + \ldots \]

\( y_1 \) and \( y_2 \) are linearly independent solutions to a 2nd order DE, so by definition, the general solution is:

\[ y = c_0 y_0 + c_1 y_1 \]

which is what we found in (a).
3. (a) Find a general solution for $ay'' + xy' + y = 0$.

\[
0 = ay'' + xy' + y = a \sum_{n=2}^{\infty} \frac{n(n-1)c_n}{n!} x^{n-2} + x \sum_{n=1}^{\infty} \frac{c_n}{n!} x^{n-1} + \sum_{n=0}^{\infty} c_n x^n
\]

\[
= \sum_{n=2}^{\infty} \frac{2n(n-1)}{n!} c_n x^{n-2} + \sum_{n=1}^{\infty} \frac{c_n}{n!} x^n + \sum_{n=0}^{\infty} c_n x^n
\]

\[
= \sum_{k=0}^{\infty} 2(k+1)(k+2)c_{k+2} x^k + \sum_{k=1}^{\infty} \frac{c_k}{k} x^k + \sum_{k=0}^{\infty} c_k x^k
\]

\[
= 2(3)(4)c_2 + c_0 + \sum_{k=1}^{\infty} \left[ \frac{2(k+2)(k+1)}{a(k+2)} \right] c_{k+2} + c_k x^k
\]

\[
\begin{align*}
\text{If } & c_0 + 4c_2 = 0 \quad \Rightarrow \quad c_2 = -\frac{c_0}{4}, \\
\text{and } & c_{k+2} = -\frac{(k+1)c_k}{2(k+2)(k+1)} = -\frac{c_k}{2a(k+2)} \\
\text{then } & c_3 = -\frac{c_1}{2a(3)} = -\frac{c_1}{6}
\end{align*}
\]

\[
\therefore \quad y = c_0 + c_1 x + \left( \frac{c_0}{4} \right) x^2 + \left( \frac{c_1}{6} \right) x^3 + \ldots
\]

(b) Solve the IVP: $ay'' + xy' + y = 0$, $y(0) = 1$, $y'(0) = 0$.

\[
y(0) = 1 \quad \Rightarrow \quad \left. c_n x^n \right|_{x=0} = 1 \quad \Rightarrow \quad c_0 = 1.
\]

\[
y'(0) = 0 \quad \Rightarrow \quad \left. c_n n x^{n-1} \right|_{x=0} = 0 \quad \Rightarrow \quad c_1 = 0.
\]

\[
\therefore \quad y = 1 - \frac{1}{4} x^2 + \ldots
\]