Math 2203 - Tutorial #2

1. Solve the following DE's.

a) \( e^{2x} y' = e^{-y} x \).

\[
\int e^{2x} y' \, dx = \int e^{-y} x \, dx \quad \Rightarrow \quad \int \frac{dy}{y} = \int \frac{x}{e^y} \, dx
\]

Let \( u = e^y \), then \( du = e^y \, dy \).

\[
\int \frac{1}{u} \, du = \int \frac{x}{u} \, du \quad \Rightarrow \quad \ln(u) = \ln(xe^y) + C
\]

\[
y = \ln(\frac{1}{2} e^{2x} + C).
\]

b) \( \frac{1}{xy} = y' e^{2x} \).

\[
y' = xe
\]

\[
\int dy = \int xe \, dx \quad \Rightarrow \quad \int \, dy = \int xe \, dx
\]

\[
\frac{1}{2} y^2 = uv - \frac{1}{2} ydv = xe dx
\]

\[
\frac{1}{2} y^2 = \frac{1}{2} xe^2 - \frac{1}{2} xe^{2x} + C
\]

\[
y^2 = xe^2 - \frac{1}{2} xe^{2x} + C
\]

\[
y = xe^{x/2} - \frac{1}{2} xe^{2x/2} + C.
\]

Implicit solution

Should ask yourself:

Is there a substitution I can make so that after I make the substitution, everything under the integral will be in terms of \( u \)?

No substitution I can make to get everything in terms of \( u \). E.g., I could try \( u = 2x \), but then \( \frac{1}{2} du = dx \), and I would still have an \( x \) under \( I \).

Use integration by parts.

Integration by parts

\[
u = x, \quad v = \frac{1}{2} e^{2x} \]

\[
\frac{du}{dx} = dx, \quad \frac{dv}{dx} = e^{2x} \, dx
\]
c) \( y' = e^{x^2} \)

\[ \int dy = \int e^{x^2} \, dx \]

No way to integrate this by hand. The anti-derivative is not an "elementary function".

\[ y = \int e^{x^2} \, dx \]

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2. Sketch the solution curves of \( y' = y^2 - y^3 \).

- Critical points:
  - The zeros of \( f(y) \):
    \[ 0 = y^2 - y^3 = y^2(1 - y) \iff y = 0 \text{ or } y = 1. \]

- Phase Portrait:

```
  y
  |
  |
  |
  |
  |
  y = 0
  y = 1
```

```
-1  0  1/2  1  2

+ + + - -
```

"1" stable (attractor).
"0" semi-stable.

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b) Create an IVP involving this DE, s.t. in the long term the solution to the IVP approaches zero.

Can see from our sketch that for \( y < 0 \) the
Solution curves approach zero as \( x \to \infty \).

\[ y' = y^2 - y^2, \quad y(0) = -1 \] would work.

(Could choose any negative value here)

**Logistic Eq:** Models simple population growth.

- Suppose a population grows proportional to its size:

\[
\frac{dp}{dt} = KP, \quad K > 0.
\]

**Example:**
- Bacteria growth
- Simple investment with compound interest

Solving this DE:

\[
\int p \, dp = K \int dt \implies \ln p = Kt + c
\]

\[
p = P_0 e^{Kt}. \quad \star \text{Exponential growth} \star
\]

- Similarly, \( \frac{dp}{dt} = -KP, \quad K > 0 \) is \( \star \text{Exponential decay} \star \)

**Example:**
- Radioactive decay
- Breakdown of a chemical

In these 2 simple models, the growth (decay) is unbounded.

Suppose an environment can sustain no more than \( P \) individuals in its population.

When \( P > P \), we want \( \frac{dp}{dt} < 0 \). The DE that models this is:

\[
\frac{dp}{dt} = KP_0 (P - P).
\]

* Make substitution \( \frac{p}{P} = \beta, \quad K = \delta. \star \)
\[ \frac{dP}{dt} = P\left( a - bP \right) \] \[ \text{Logistic Equation} \]

Solving the Logistic Eqn: It's autonomous \( \Rightarrow \) separable.

[All autonomous DE's are separable]

\[ \int dt = \int \frac{1}{P(a-bP)} \, dP \]

[to integrate this we can use partial fractions]

\[ \frac{1}{P(a-bP)} = \frac{A}{P} + \frac{B}{a-bP} = \frac{A(a-bP) + BP}{P(a-bP)} \]

\[ aA = 1 + -Ab + B = 0 \]

\[ A = \frac{a}{a} \quad B = \frac{b}{a} \]

So, \[ \frac{1}{P(a-bP)} = \frac{1}{a} + \frac{b/a}{a-bP} \]

\[ \therefore \int dt = \int \left( \frac{1}{a} + \frac{b/a}{a-bP} \right) \, dP \]

\[ \therefore t = \frac{1}{a} \ln|P| + \frac{b}{a} \ln|a-bP| + C \]

\[ \therefore t + C = \frac{1}{a} \left[ \ln|P| - \ln|a-bP| \right] \]

\[ \therefore a(t+C) = \ln |P/a-bP| \Rightarrow C \cdot e = \frac{p}{a-bP} \]

\[ \therefore a \cdot e^t - b \cdot C \cdot e^t = P = P \Rightarrow P = \frac{C \cdot e^t}{1 + b \cdot e^t} \]
Suppose \( P(0) = P_0 \). \( P_0 \neq \frac{a}{b} \).

Substituting this in and solving for \( c_1 \):

\[
P_0 = \frac{ac_1}{1 + bc_1} \quad \Rightarrow \quad P_0 + bc_1P_0 = ac_1 \quad \Rightarrow \quad P_0 = (a - bP_0)c_1
\]

\[
\Rightarrow \quad c_1 = \frac{P_0}{a - bP_0}.
\]

Substituting this back into the eqn:

\[
P = aP_0 \frac{a - bP_0}{a - bP_0 - \frac{aP_0}{a - bP_0}} = \frac{aP_0}{bP_0 + (a - bP_0)e^{-at}}.
\]

From the solution, we can see that as \( t \to \infty \)

\[
\lim_{t \to \infty} \frac{P}{P_0} = \frac{a}{b}.
\]

As \( t \to -\infty \), \( P \to 0 \).

Sketch of solution curves.

\[a = 2, \quad b = 1\]:

\[
P = \frac{2P_0}{P_0 + (2 - P_0)e^{-at}}.
\]
3. Suppose that the population \( P \) (in thousands) of squirrels in Hamilton can be modelled by the DE \( \frac{dP}{dt} = P(2-P) \).

@ Is the initial population of squirrels is 3000, what can you say about the long-term behaviour of the squirrel population?

\[ P(2-P) = 0 \quad \Rightarrow \quad P = 0 \quad \text{or} \quad P = 2. \]

\[ \text{Squirrel population will tend to 2000 in the long-term.} \]

@ Can a population of 1000 ever decline to 500? Explain.

No, b/c solution curves strictly increasing b/w 0 \& P=2, so if we begin w/ 1000 it'll never decrease to 500.

@ Can a population of 1000 ever increase to 3000? Explain.

No, b/c a solution curve starting at \( P=1 \) can't cross the constant solution \( P=2 \), so it will tend to 2000, but can never reach 3000.
4. Consider the IVP \( y' = 2x - 3y + 1 \), \( y(1) = 5 \).
Find an approximation of \( y(1.1) \) using Euler's method with a step size of \( h = 0.1 \).

\[
y_{n+1} = y_n + \frac{h}{10} F(x_n, y_n), \quad x_n = x_0 + nh.
\]
Here \( F(x, y) = 2x - 3y + 1 \). \( x_0 = 1 \), \( y_0 = 5 \).

\( n = 1 \):
\[
y_1 = y_0 + \frac{h}{10} F(x_0, y_0) = 5 + \frac{1}{10} F(1, 5)
= 5 + \frac{1}{10} (2 - 15 + 1) = 5 + \frac{1}{10} (-12) = \frac{50}{10} - \frac{12}{10}
= \frac{38}{10} = \frac{19}{5}. \quad x_1 = x_0 + h = 1 + \frac{1}{10} = \frac{11}{10}.
\]

\( (x_1, y_1) = (\frac{11}{10}, \frac{19}{5}) \).

\( n = 2 \):
\[
x_2 = x_0 + 2 \left(\frac{1}{10}\right) = \frac{10}{10} + \frac{2}{10} = \frac{12}{10} = 1.2.
\]
\[
y_2 = y_1 + \frac{h}{10} F(x_1, y_1) = \frac{19}{5} + \frac{1}{10} \left( \frac{11}{5} - \frac{53}{5} + 1 \right)
= \frac{19}{5} + \frac{1}{10} \left( \frac{11}{5} - \frac{53}{5} + \frac{5}{5} \right)
= \frac{19}{5} + \frac{1}{10} \left( \frac{-41}{5} \right)
= \frac{190}{50} - \frac{41}{50} = \frac{149}{50} = 2.98. \quad (x_2, y_2) = (1.2, 2.98).
\]

\( \therefore y(1.1) \approx 2.98 \).