1. Suppose that the population \( p \) (in thousands) of squirrels in Hamilton can be modelled by the DE \( \frac{dp}{dt} = p(2-p) \).

@ If the initial population of squirrels is 3000, what can you say about the long-term behaviour of the squirrel population?

\[ p(2-p) > 0 \, \Rightarrow \, p = 0 \text{ or } p = 2. \]

\[ p' \]

\[ p(0) \]

\[ p(2) \]

\[ p(1) \]

\[ p(3) \]

Squirrel population will tend to 2000 in the long-term.

@ Can a population of 1000 ever decline to 500? Explain.

No, b/c solution curves strictly increasing b/w 0 \& 2, so if we begin w/ 1000 it'll never decrease to 500.

@ Can a population of 1000 ever increase to 3000? Explain.

No, b/c a solution curve starting at \( p = 1 \) can't cross the constant solution \( p = 2 \), so it will tend to 2000, but can never reach 3000.
How many squirrels will there be after one year if the initial population of squirrels is 50?

\[ p' = p(2-p) \]

\[ \int \frac{1}{p(2-p)} \, dp = \int -\frac{1}{a} \, dt \]

\[ \frac{1}{p} - \frac{1}{2} \ln(2-p) = -\frac{t}{a} + c \]

\[ \frac{1}{2} \ln(2-p) = \frac{1}{2} \ln a - \frac{t}{a} + c \]

\[ \ln \left| \frac{p}{2-p} \right| = -\frac{t}{a} + c \]

\[ p = \frac{ce^{at}}{1 + ce^{at}} \]

Therefore, 50 squirrels = \(\frac{1}{20}\) thousand squirrels.

So, \( p(0) = \frac{1}{20} \).

\[ \frac{1}{a} = \frac{2c}{1+c} \Rightarrow \frac{1}{20} = \frac{2c}{1+c} \Rightarrow \frac{1}{a} = \frac{34}{20} \Rightarrow c = \frac{1}{34}. \]

So \( p = \frac{2e^{at}}{34(1 + \frac{1}{34} e^{at})} = \frac{2e^{at}}{34 + e^{at}} \).
\[ P(1) = \frac{2e^{2}}{39 + e^{2}} \]

So, after one year, there will be
\[ \frac{2e^{2}}{39 + e^{2}} \approx 0.3196 \text{ thousand squirrels} \]
\[ \approx 319 \text{ squirrels} \]

\[ 98 + (9-6)A = 1 \]
\[ A = \frac{1}{9-6} = \frac{1}{3} \]

\[ 0 = 8 + A - 6 \]
\[ A = 1 \]
\[ A + 2c = 1 \]
\[ c = \frac{1}{2} \]
\[ c + \frac{1}{2} = 1 - \frac{1}{9-6} = \frac{1}{3} \]
\[ c = \frac{1}{3} \]

\[ 6 = 9c + f \]
\[ f = \frac{6}{11} \]

\[ \frac{1}{f} = \frac{11}{6} \]

\[ b = \frac{1}{f} = \frac{11}{6} \]

\[ x = 30 \]

\[ y = 30 \]

\[ x + y = 60 \]
2. Consider the IVP \( y' = 2x - 3y + 1 \), \( y(1) = 5 \). Find an approximation of \( y(1.1) \) using Euler's method with a step size of \( h = 0.1 \).

\[
y_{n+1} = y_n + \frac{h}{10} F(x_n, y_n), \quad x_n = x_0 + nh.
\]

Here \( F(x, y) = 2x - 3y + 1 \). \( x_0 = 1 \), \( y_0 = 5 \).

\( n = 1 \):

\[
y_1 = y_0 + \frac{h}{10} F(x_0, y_0) = 5 + \frac{1}{10} F(1, 5)
\]

\[
= 5 + \frac{1}{10} (2 - 15 + 1) = 5 + \frac{1}{10} (-12) = \frac{50}{10} - \frac{12}{10}
\]

\[
= \frac{38}{10} = \frac{19}{5}. \quad \text{So, } x_1 = x_0 + h = 1 + \frac{1}{10} = \frac{11}{10}.
\]

\[
(x_1, y_1) = \left( \frac{11}{10}, \frac{19}{5} \right).
\]

\( n = 2 \):

\[
x_2 = x_0 + 2 \left( \frac{1}{10} \right) = \frac{10}{10} + \frac{2}{10} = \frac{12}{10} = 1.2.
\]

\[
y_2 = y_1 + \frac{h}{10} F(x_1, y_1) = \frac{19}{5} + \frac{1}{10} \left( \frac{11}{5} - 5 \frac{3}{5} + 1 \right)
\]

\[
= \frac{19}{5} + \frac{1}{10} \left( \frac{11}{5} - \frac{15}{5} \right) = \frac{19}{5} + \frac{1}{10} (-\frac{41}{5})
\]

\[
= \frac{190}{50} - \frac{41}{50} = \frac{149}{50} = 2.98. \quad (x_2, y_2) = (1.2, 2.98).
\]

\[
\therefore y(1.2) \approx 2.98.
\]
3. Solve \( xy' - y = 2x \ln x \).

\[
y' - \frac{5}{x} y = 2x \ln x
\]

\[
\int \left( \frac{5}{x} \right) \, dx = \int \frac{5}{x} \, dx = 5 \ln x.
\]

\[
y = e^{\int \frac{5}{x} \, dx} = e^{5 \ln x} = x^5.
\]

\[
\int \left[ e^{-5 \ln x} (2x \ln x) \right] \, dx
\]

\[
x \int \left[ e^{-5 \ln x} \right] \, dx = x e^{5 \ln x} = xe^{5 \ln x}.
\]

\[
\int \left[ 2x \ln x \right] \, dx = 2 \int \ln x \, dx = 2 \left( \frac{1}{2} x \ln x - \frac{1}{2} x \right) = x (\ln x)^2 - c x.
\]

\[
y = x (\ln x)^2 + c x.
\]

b) Find the largest interval where this solution is defined.

Recall: Need to find an interval where \( y \) is \( C^1 \) (continuously differentiable).

\[
y \text{ defined for } x > 0. \quad y(x) = \ln x.
\]

\[
y' = (\ln x)^2 + \frac{2x (\ln x)}{x} + c \text{ cont. on } (0, \infty).
\]

\[
\therefore \text{ solution defined on } (0, \infty).
\]
4. A large tank is partially filled with 100 gallons of fluid in which 10 pounds of salt is dissolved. Brine containing \( \frac{1}{2} \) pound of salt per gallon is pumped into the tank at a rate of 6 gal/min. The well-mixed solution is then pumped out at a slower rate of 4 gal/min. Find the number of pounds of salt in the tank after 30 minutes.

Let \( A(t) \) denote the amount of salt in the tank at time \( t \) (measured in lb).

Want to find \( A(30) \).

We know \( A(0) = 10 \) lb.

We also know:

\[
\frac{dA}{dt} = \text{(input rate of salt)} - \text{(output rate of salt)}
\]

\[
\text{Rin} = \left( \text{Concentration of salt inflow} \right) \times \left( \frac{\text{input rate}}{\text{gal/min}} \right) = \frac{1}{2} \text{ lb/gal} \times 6 \text{ gal/min} = 3 \text{ lb/min}.
\]

So, \( \text{Rin} = 3 \text{ lb/min} \).
\[
R_{\text{out}} = \left( \text{Concentration of salt outflow} \right) \times \left( \text{Output rate of brine} \right) = \left( \text{Output rate of salt} \right)
\]

\[
\frac{A}{\text{Amount of brine}} = 4 \text{ gal/min}
\]

To find this, we need to know how much brine is in the tank at time \( t \).

The liquid accumulates in the tank at the rate of \( \text{Fin} - \text{Fout} = 6 \text{ gal/min} - 4 \text{ gal/min} = 2 \text{ gal/min} \).

\[\therefore \text{After } t \text{ minutes there are } 100 + 2t \text{ gallons of brine in the tank.}\]

So, the concentration of outflow is \( \frac{A}{100 + 2t} \).

\[\therefore R_{\text{out}} = \left( \frac{A}{100 + 2t} \right) \times (4 \text{ gal/min}) = \frac{4A}{100 + 2t} \text{ lb/min}.\]

\[\therefore \frac{dA}{dt} = 3 - \frac{4A}{100 + 2t}\]

\[\therefore A + \frac{4}{100 + 2t} = 3.\]

\[\therefore A = e^{\int \frac{4}{100 + 2t} dt} = 2 \ln(100 + 2t) + C.\]

\[\text{Linear}
\]

\[\int px dx = \frac{p}{2} \ln|100 + 2t| + C.
\]

\[A = 2 \ln(100 + 2t) + C \text{ as } u = 2t + 100, \quad \frac{du}{2} = dt.
\]

\[A = \frac{3}{2} \left( 2t + 100 \right)^{3} + C.
\]
\[ A(0) = 10 \]

\[ 10 = 50 + \frac{c}{100} \Rightarrow c = -40 \cdot 10000 = -400000. \]

\[ \therefore A = 50 + \frac{c}{100} - 400000 \cdot (2t + 100) \]

\[ \therefore A(30) = 50 - 400000(160) = \frac{515}{8} = 64.375. \]

After 30 min, there will be \( \approx 64.38 \) lb of salt in the tank.