Consider $A = \begin{bmatrix} -1 & 2 \\ 0 & -3 \end{bmatrix}$. Find $A^k$ using the Cayley-Hamilton Theorem.

$0 = \begin{vmatrix} -1-\lambda & 2 \\ 0 & -3-\lambda \end{vmatrix} = (-1-\lambda)(-3-\lambda) \Rightarrow \lambda = -1, \lambda = -3.$

Characteristic Eqn: $\lambda^2 + 4\lambda + 3 = 0$.

By Cayley-Hamilton Theorem, $A^2 + 4A + 3I = 0$, so it's always possible to write $A^k = c_0 I + c_1 A$ for some constants $c_1, c_2, \ldots, A^k = c_0 + c_1 A$.

$A^k = c_0 + c_1 A$ $\Rightarrow$ $(-3)^k = c_0 + c_1 (-3)$ $\Rightarrow$ $c_0 - 3c_1 = (-3)^k$ $\Rightarrow$ $c_0 - c_1 = (-1)^k$.

$c_0 = (-1)^k + c_1$ $\Rightarrow$ $c_0 = -\frac{1}{2} (-3)^k + \frac{3}{2} (-1)^k$. $-2c_1 = (-3)^k - (-1)^k$.

$c_1 = -\frac{1}{2} (-3)^k + \frac{1}{2} (-1)^k$.

$S_0, \ A^k = c_0 I + c_1 A = \begin{bmatrix} -\frac{1}{2} (-3)^k + \frac{3}{2} (-1)^k & 0 \\ 0 & -\frac{1}{2} (-3)^k + \frac{1}{2} (-1)^k \end{bmatrix} + c_1 \begin{bmatrix} -1 & 2 \\ 0 & -3 \end{bmatrix} = \begin{bmatrix} (-1)^k & (-3)^k + (-1)^k \\ 0 & (-3)^k \end{bmatrix}$. 
2. Suppose a shopkeeper wants to put up a rectangular sign of length $L$ for his store so that the deflection of the sign can be modelled by the 4th-order DE $EI y''''(x) = w(x)$. Identify the appropriate boundary conditions:

- **a** He uses one nail on each side.
  - Hinged on both sides: $y(0) = 0$, $y'(0) = 0$, $y''(L) = 0$, $y'''(L) = 0$.

- **b** He uses 2 nails on each side.
  - Embedded on both sides: $y(0) = 0$, $y'(0) = 0$, $y''(L) = 0$, $y'''(L) = 0$.

- **c** He uses 3 nails on the left & no nails on the right.
  - Embedded on left & free on right.
  - $y(0) = 0$, $y''(L) = 0$, $y'(0) = 0$, $y'''(L) = 0$.

- **d** He uses 2 nails on left & a stack of crates on right.
  - Embedded on left & simply supported on right.
  - $y(0) = 0$, $y(L) = 0$, $y'(0) = 0$, $y''(L) = 0$. 


3. Find the deflection, \( y(x) \), in \( [0, L] \) if \( w(x) = w_0 \) a constant \( 0 < x < L \).

\[
y^{(4)} = \frac{w_0}{EI}, \quad y(0) = 0, \quad y'(0) = 0, \quad y(L) = 0, \quad y''(L) = 0.
\]

\[m = 0, \quad m = 0 \text{ w/ multiple y.}
\]

\[y_c = c_1 + c_2 x + c_3 x^2 + c_4 x^3.
\]

\[y_p = A x^4 \quad y^{(4)}_p = \frac{w_0}{EI} \quad \Rightarrow 24A = \frac{w_0}{EI} \quad \Rightarrow A = \frac{w_0}{24EI}.
\]

\[y_p = 4A x^3 \quad y^{(3)}_p = 12A x^2 \quad y^{(2)}_p = 24A x \quad y''_p = 24A.
\]

\[y(0) = 0 \Rightarrow 0 = c_1 \quad y(L) = 0 \Rightarrow c_3 L^2 + c_4 L + \frac{w_0 L^4}{24EI} = 0
\]

\[y'(0) = 0 \Rightarrow 0 = c_2 \quad y''(L) = 0 \Rightarrow 2c_3 + 6c_4 L + \frac{w_0 L^2}{24EI} = 0
\]

\[c_3 = -c_4 L - \frac{w_0 L^2}{24EI}.
\]

\[c_4 = -\frac{w_0}{12EI} L - \frac{3c_3}{3L} = -\frac{w_0}{12EI} L + c_4 + \frac{w_0 L^2}{24EI}.
\]

\[c_3 = -\frac{w_0}{72EI} L - \frac{3L}{3} \quad c_4 = -\frac{5w_0}{48EI} L = \frac{c_3}{24EI} \quad \frac{5w_0 L^2}{144EI}.
\]

\[\therefore y = \frac{w_0}{16EI} L^2 x^2 - \frac{5}{48} \frac{w_0}{EI} L x^3 + \frac{w_0}{24EI} x^4
\]

\[= \frac{w_0}{48EI} \left[ 3L^2 x^2 - 5L x^3 + 2x^4 \right].
\]
4. Solve the homog. system of linear DE's:

\[ X' = \begin{bmatrix} 6 & -1 \\ 5 & 4 \end{bmatrix} X, \quad X(0) = \begin{bmatrix} -2 \\ 8 \end{bmatrix}. \]

\[ 0 = \begin{bmatrix} 6 - \lambda & -1 \\ 5 & 4 - \lambda \end{bmatrix} = 2\lambda - 10\lambda + \lambda^2 + 5 = \lambda^2 - 10\lambda + 29. \]

\[ \lambda = \frac{10 \pm \sqrt{100 - 116}}{2} = 5 \pm 2i. \]

\[ \lambda = 5 + 2i: \]

\[ \begin{bmatrix} 6 - 5 - 2i & -1 \\ 5 & 4 - 5 - 2i \end{bmatrix} = \begin{bmatrix} 1 - 2i & -1 \\ 5 & 1 - 2i \end{bmatrix} \]

\[ y = (1 - 2i)x = (1 - 2i) \begin{bmatrix} 1 \\ 1 - 2i \end{bmatrix} = \begin{bmatrix} 1 \\ 1 - 2i \end{bmatrix} \]

\[ X = c_1 \begin{bmatrix} 1 \\ 1 - 2i \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ -2 \end{bmatrix}. \]

\[ X(0) = \begin{bmatrix} -2 \\ 8 \end{bmatrix} \Rightarrow c_1 \begin{bmatrix} 1 \\ 1 - 2i \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ -2 \end{bmatrix} = \begin{bmatrix} -2 \\ 8 \end{bmatrix}. \]

\[ \begin{bmatrix} 1 & 0 & -2 \\ 1 & -2i & 8 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -2 \\ 0 & -2i & 10 \end{bmatrix} \]

\[ c_1 = -2, \quad c_2 = -10 \Rightarrow c_2 = -5. \]

\[ X = -2e^{5t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \cos(2t) - \begin{bmatrix} 0 \\ -2 \end{bmatrix} \sin(2t) - 5e^{5t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \sin(2t) + \begin{bmatrix} 0 \\ -2 \end{bmatrix} \cos(2t). \]
\[
\begin{bmatrix}
-2 e^{5t} \cos(at) - 5 e^{5t} \sin(at) \\
-2 e^{5t} \cos(at) - 2 e^{5t} \sin(at) - 5 e^{5t} \sin(at) + 4 e^{5t} \cos(at)
\end{bmatrix}
\]

\[
= \begin{bmatrix}
5 e^{5t} (2 \cos(at) + 5 \sin(at)) \\
5 e^{5t} (8 \cos(at) - 9 \sin(at))
\end{bmatrix}
\]

b) Sketch the solution curve corresponding to this 3D.

Complex eigenvalues $\lambda = 5 \pm 2i$.

\[a70: \text{We will have a spiral moving away from origin, passing through } (x(0), y(0)) = (\text{-2, 8}).\]
\[
\begin{align*}
\left( \cos \theta e^{j\phi} \right)^4 &= (\cos^4 \theta + 6 \cos^2 \theta \sin^2 \theta + \sin^4 \theta) e^{j4\phi} \\
&= \left( \frac{1 + \cos 2\phi}{2} \right)^2 e^{j4\phi} \\
&= \frac{1}{4} (1 + 2 \cos 2\phi + \cos 4\phi) e^{j4\phi}
\end{align*}
\]