Mat 202 - Tutorial #1

1. Count the number of n-digit binary sequences with exactly r 1's (i.e. n-r 0's).

Recall: The number of r-subsets of an n-set is \( \binom{n}{r} = \frac{n!}{r!(n-r)!} \).

We can think about having n slots:

\[
\begin{array}{ccccccc}
\hline
1 & 2 & 3 & \ldots & n-1 & n \\
\hline
\end{array}
\]

We need r slots to be 1 and n-r slots to be 0. How many ways are there to choose r slots?

\[ \text{There are } \binom{n}{r} \text{ ways.} \]

\[ \therefore \text{There are } \binom{n}{r} \text{ different ways we can place the 1's in r slots } \Rightarrow \text{there are } \binom{n}{r} \text{ many such binary sequences.} \]
2. How many ways are there to seat 6 different boys and 6 different girls along one side of a long table with 12 seats?

b. How many ways if boys and girls alternate seats?

\[ \text{Recall: The number of arrangements of } K \text{ distinct elements from a set of size } N \text{ is:} \]
\[ P(K, N) = \frac{n(n-1)\ldots(n-K+1)}{(n-K)!} = \frac{n!}{(n-K)!} . \]

\[ \text{a. We have 12 people and we want to seat them in 12 seats:} \]
\[ \frac{12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10 \cdot 11 \cdot 12} = 12! \]

12! ways!

\[ \text{i.e. } P(12, 12) = \frac{12!}{(12-12)!} = \frac{12!}{1} = 12! . \]

\[ \approx 479,001,600 \]

\[ \text{b. Either all boys sit in odd seats or all boys sit in even seats.} \]

There are \( P(6,6) = 6! \) ways to sit the boys in the 6 odd seats. There are \( P(6,6) = 6! \) ways to sit the girls in the 6 even seats.

\[ \therefore \text{There are } 2 \cdot 6! \cdot 6! \text{ ways.} \]

\[ \approx 1,036,800 \]}
3. How many arrangements are there of the 7 letters in the word "UNUSUAL"?

First notice that there are 3 U's, 1 of each other letter.

- How many ways are there to position the U's?

  1 2 3 4 5 6 7

  There are \( \binom{7}{3} \) ways.

(Same letter so order doesn't matter...)

- Once we position the U's, how many ways can we arrange the other letters?

  Once the U's are placed there are 4 spots left.

  \[ \cdot: \text{P}(4, 4) = \frac{4!}{0!} = 4! \text{ ways.} \]

  \[ \cdot: \text{There are} \left(\frac{7!}{4!3!}\right)(4!) = \frac{7!}{3!} \text{ ways.} \]

Note: You could also think about this in the following way:

7 letters can be arranged 7! ways. Since there are 3 U's, you can rearrange them in the word 3! ways without changing the word. \( \cdot: \text{It's} \ \frac{7!}{3!} \).
6. How many ways can you arrange the letters in the word MISSISSIPPI.

1 M
4 I's
4 S's
2 P's

(11) ways to arrange the I's.
(4) ways to arrange the S's. Once the I's have been placed.
(7) ways to arrange the P's. Once the I's and S's have been placed.
(1) ways to arrange M, once the other letters have been placed.

\[(\frac{11!}{4!4!2!1!}) = \frac{11!}{7!4!4!3!2!1!1!} = \frac{11!}{4!3!2!1!1!} = \frac{11!}{4!3!2!} = 34,650\] ways.
4. What is the probability that a 5-card poker hand has the following?

a) Four Aces

Recall: The probability that an event will occur is:

\[
\frac{\text{# of ways an event can occur}}{\text{# of possible outcomes}}
\]

Here, a deck has 52 cards, therefore there are

\[
\binom{52}{5} = \frac{52!}{5!47!} = 2,598,960 \text{ possible poker hands.}
\]

* How many ways can we choose 4 Aces?

\[
\binom{4}{4} = 1 \text{ way.}
\]

* Once we have 4 Aces, how many options do we have for the 5th card?

\[
\binom{48}{1} = 48
\]

\[
\therefore \text{There are 48 different poker hands with 4 aces.}
\]

\[
\therefore \text{The probability is} \quad \frac{48}{\binom{52}{5}} = \frac{1}{54,145}.
\]

b) Four of a Kind.

By a), there are 48 different poker hands with 4 cards of the same suit. There are 13 different ranks (types of cards).

\[
\therefore \text{There are } 13 \times 48 = 624 \text{ such hands.} \quad \text{The probability is}
\]

\[
\frac{624}{\binom{52}{5}} = \frac{1}{4165}.
\]
A Full house. [i.e. 3 of one suit and 2 of a different suit].

• How many ways to choose 3 of one rank?

\( \binom{13}{1} \) ways to choose one rank.

\( \binom{4}{3} \) ways to choose 3 cards from the rank we chose.

\[ \therefore \binom{13}{1} \binom{4}{3} \text{ ways.} \]

• How many ways to choose 2 of a different rank?

\( \binom{12}{1} \) ways to choose one rank.

\( \binom{4}{2} \) ways to choose 2 cards from the rank we chose.

\[ \therefore \binom{12}{1} \binom{4}{2} \text{ ways.} \]

\[ \therefore \text{There are } \binom{13}{1} \binom{4}{3} \binom{12}{1} \binom{4}{2} \text{ different Full house poker hands.} \]

\[ 13 \cdot \frac{4!}{3!} \cdot 12 \cdot \frac{4!}{2!2!} = 3744 \]

\[ \therefore \text{The probability is } \frac{3744}{\binom{52}{5}} = \frac{6}{4165}. \]
5. In a race with 10 horses, the 1st, 2nd, and 3rd place finishers are noted. How many outcomes are there?

\[ P(3, 10) = \frac{10!}{(10-3)!} = 10 	imes 9 	imes 8 = 720. \]

6. A bridge hand consists of 13 cards.
   a) How many bridge hands are there?

\[ \binom{52}{13} = 635,013,559,600. \]

b) What is the probability that a bridge hand has 4 spades (and 9 cards that are not spades)?

13 spades. \( \binom{13}{4} \) is the number of subsets of spades of size 4.

39 non-spades. \( \binom{39}{9} \) subsets of non-spades of size 9.

\[ \therefore \binom{13}{4} \binom{39}{9} \text{ possible hands.} \]

Probability is:

\[ \frac{\binom{13}{4} \binom{39}{9}}{\binom{52}{13}} \approx 0.2386. \]
What is the probability that a bridge hand has 4 spades and 3 hearts (i.e., 7 remaining cards are diamonds and clubs):

\[
\begin{array}{ccc}
\text{spades} & \text{hearts} & \text{diamonds/clubs} \\
13 & 13 & 26 \\
4 & 3 & 6 \\
\hline
52 & & 13 \\
\end{array}
\]

\[\approx 0.0741.\]