Pigeonhole Principle: Placing more than $Kn$ objects into $n$ classes cuts more than $k$ objects into some class.

i.e.] Given a set of $m$ objects and $n$ groups to place them in with $n < m$, then one group must have \( \left\lfloor \frac{m-1}{n} \right\rfloor + 1 \) objects in it.

1. Given 14 people, each can receive a grade of A, B, or C.

[a] How many people must share a grade?

Here $m=14$, $n=3$.

$$ \left\lfloor \frac{m-1}{n} \right\rfloor + 1 = \left\lfloor \frac{13}{3} \right\rfloor + 1 = 4 + 1 = 5. $$

At least 5 people must share a grade.

[b] How many people would be needed in order for the Pigeonhole Principle to guarantee that 9 people share a grade?

$$ \left\lfloor \frac{m-1}{3} \right\rfloor + 1 = 9 \iff \left\lfloor \frac{m-1}{3} \right\rfloor = 8 \iff m-1 = 24, 25, 26 \iff m = 25, 26, 27. $$

Therefore, at least 25 people would be needed.
C) How many people to ensure 2 people get an A?

No number suffices since the Pigeonhole Principle cannot force pigeons into a specific hole.

2. Suppose you are given 27 different numbers between 15 and 60 (inclusive). Show that some pair differ by exactly 6.

Let's denote our 27 numbers by $x_1, x_2, \ldots, x_{27}$.

We want to show that there exists an $1 \leq i, j \leq 27$ such that $x_j - x_i = 6 \Rightarrow x_j = x_i + 6$.

Consider the two sets

15 $\leq x_1 < x_2 < \ldots < x_{27}$ $\leq 60$

21 $\leq x_1 + 6 < x_2 + 6 < \ldots < x_{27} + 6$ $\leq 66$.

We wish to show that these two sets have an element in common.

There are at most 66 - 15 + 1 = 52 distinct integers in these 2 sets. But there are 27 + 27 = 54 elements in the 2 sets.

\[
\left\lfloor \frac{54 - 1}{52} \right\rfloor + 1 = 1 + 1 = 2. \text{ By Pigeonhole Principle,}
\]

at least 2 elements in these sets must be the same integer $\exists i \neq j$ s.t. $x_j = x_i + 6 \Rightarrow x_j - x_i = 6$.\]
3. Suppose 51 points are chosen at random inside a 1x1 square. Prove that 3 of these points are inside of a circle of radius $\frac{1}{\sqrt{7}}$.

Subdivide the box into 25 sub boxes of length $\frac{1}{5}$.

\[
\left\lfloor \frac{51-1}{25} \right\rfloor + 1 = 2 + 1 = 3.
\]

There must exist at least one sub box which contains 3 points.

The circle circumscribed around the square has radius $\sqrt{\left(\frac{1}{5}\right)^2 + \left(\frac{1}{5}\right)^2} = \sqrt{\frac{2}{25}} = \sqrt{\frac{1}{7}} < \sqrt{\frac{1}{7}} \approx \frac{1}{\sqrt{7}}$.

So, this circle sits inside of a circle of radius $\frac{1}{\sqrt{7}}$ and the 3 points lie in a circle of radius $\frac{1}{\sqrt{7}}$. 
4. [#3] How many nonempty collections of letters can be formed from 4 A's and 8 B's?

Want \((A,B)\) whose have \(0 \leq A \leq 4, 0 \leq B \leq 8\), and want to exclude \((0,0)\). \(\text{E.g.: } (2,3) \leftrightarrow \text{AAAABB} \).

5 choices for A and 9 choices for B:

5 \cdot 9 - 1 = 45 - 1 = 44 nonempty collections.