Recall: A relation b/w sets S & T is a subset of S x T. A relation on S is a subset of S x S.

- An equivalence relation on a set S is a relation R on S s.t. \( \forall x,y,z \in S: \)
  - Reflexive: \( (x,x) \in R \) \( (x=x) \)
  - Symmetric: \( (x,y) \in R \iff (y,x) \in R \)
  - Transitive: \( (x,y) \in R \land (y,z) \in R \implies (x,z) \in R. \)

- An order relation is a relation that is reflexive, transitive, and antisymmetric \( (x,y) \in R \land (y,x) \in R \implies x=y. \)

1. Which of the following are equivalence relations?

   a) \( \equiv(a,b) \mid a \neq b \) are the same age.  
      \[
      \begin{align*}
      x \sim x &. \checkmark \\
      x \sim y \equiv y \sim x &. \checkmark \\
      x \sim y \land y \sim z \equiv x \sim z &. \checkmark \text{Yes.}
      \end{align*}
      \]

   b) \( \equiv(a,b) \mid a \neq b \) have the same parents.  
      \[
      \begin{align*}
      x \sim x &. \checkmark \\
      x \sim y \equiv y \sim x &. \checkmark \\
      x \sim x \land y \sim z \equiv x \sim z &. \checkmark \text{Yes.}
      \end{align*}
      \]

   c) \( \equiv(a,b) \mid a \neq b \) share a common parent.  
      \[
      \begin{align*}
      x \sim x &. \checkmark \\
      x \sim y \equiv y \sim x &. \checkmark \\
      \text{Not transitive though: } x \neq y \text{ could have the same mom, but different dads}.
      \end{align*}
      \]
d. \[ \{(a, b) \mid a \& b \text{ have met}\} \]

Not transitive.

e. \[ \{(a, b) \mid a \& b \text{ speak common languages}\} \]

Not transitive.

d. Which of the following are equivalence relations of order relations?

a. \[ \{(a, b) \mid a \text{ is an ancestor of } b\} \]

Suppose every individual is an ancestor of itself.

\[ x \sim x, \quad y \sim y, \quad x \sim y \quad \text{Not symmetric.} \]

\[ x \sim y \land y \sim x \iff x = y \]

\[ x \sim y \land y \sim z \iff x \sim z. \quad \checkmark \]

\[ x \sim x \quad \checkmark \]

\[ x \sim y \quad \checkmark \]

\[ x \sim z \quad \checkmark \]

\[ \therefore \text{order relation.} \]

b. \[ (x_1, y_1) \sim (x_2, y_2) \iff x_1 y_2 = x_2 y_1, \text{ where } x, y, x_1, y_1 \text{ positive integers.} \]

\[ (x_1, y_1) \sim (x_1, y_1), \quad \text{since } x_1 y_1 = y_1 x_1. \quad \checkmark \]

\[ (x_1, y_1) \sim (x_2, y_2) \iff x_1 y_2 = x_2 y_1 \iff (x_2, y_2) \sim (x_1, y_1). \quad \checkmark \]

Suppose \[ (x_1, y_1) \sim (x_2, y_2) \& (x_2, y_2) \sim (x_3, y_3) \]

\[ x_1 y_2 = x_2 y_1 \quad \land \quad x_2 y_3 = y_2 x_3. \quad \text{WTS } x_1 y_3 = x_3 y_1, \]

\[ x_2 = \frac{x_1 y_2}{y_1} \]

\[ \frac{x_2 y_3}{y_1} \]

\[ \frac{x_1 y_2 y_3}{y_1} = y_2 x_3 \iff x_1 y_3 = x_3 y_1 \iff (x_1, y_1) \sim (x_3, y_3). \quad \checkmark \]

\[ \therefore \text{Equivalence relation.} \]
3. On the set \( \mathbb{Z} \times \{1, 2, 3\} \) consider the equivalence relations:

\( R_1 = \{(1,1), (2,2), (3,3), (1,2), (2,1)\} \) and

\( R_2 = \{(1,1), (2,2), (3,3), (2,1), (1,3)\} \).

a) Is \( R_1 \cup R_2 \) an equivalence relation?

(1,2) and (2,3) are in our set, so \( 1 \cup 2 \) and \( 2 \cup 3 \).

But 1 \( \times 3 \) by (1,3) not in \( R_1 \cup R_2 \).

So \( R_1 \cup R_2 \) not have transitive property = Not an equivalence relation.

Recall: Given a set \( S \) and an equivalence relation \( R \) on \( S \), the set of elements equivalent to \( x \in S \) is the equivalence class containing \( x \).

b) List the equivalence classes of \( R_1 \). List the equivalence classes of \( R_2 \).

\( R_1 : [1J] = \{1, 2\}, [3J] = \{3\} \).

\( R_2 : [1J] = \{1\}, [2J] = \{2, 3\} \).

4. The relation \( R \) on \( \mathbb{Z} \) is defined by \( x \sim y \iff x + 3y \) is even. Prove that this is an equivalence relation. Find the equivalence classes.

- \( x \sim x \) since \( x + 3x = 4x = 2(2x) \) is even.
- \( x \sim y \iff x + 3y \) even \iff \( x = 2k - 3y \). Then \( y + 3x = y + 3(2k - 3y) \)
  \[= y + 6k - 9y = 6k - 8y = 2(3k - 4y) \] even \iff \( y \sim x \).
• Suppose \( x \sim y \) and \( y \sim z \) \(\equiv\) \( x + 3y - 2z = 2k \quad y + 3z = 2l \)

\(\equiv\) \( x = 2k - 3y + 3z = 2l - y \)

\(\equiv\) \( x + 3z = 2k - 3y + 2l - y = 2k + 2l - 4y = 2(k + l - 2y) \) even

\(\equiv\) \( x \sim z \).

\(\therefore\) Equivalence Relation.

\(\{0\} = \{x \in \mathbb{Z} \mid x \sim 0\} = \{x \in \mathbb{Z} \mid x + 3(0) \text{ even}\} = \{x \in \mathbb{Z} \mid x \text{ even}\}.

\(\{1\} = \{x \in \mathbb{Z} \mid x \sim 1\} = \{x \in \mathbb{Z} \mid x + 3(1) \text{ even}\} = \{x \in \mathbb{Z} \mid x \text{ odd}\}.

This partitions \( \mathbb{Z} \) into two distinct equivalence classes.

• Recall: Given \( n \in \mathbb{N} \), \( x, y \in \mathbb{Z} \) are congruent modulo \( n \) if \( x - y \) is divisible by \( n \), i.e., \( x \equiv y \mod n \). This gives an equivalence relation on \( \mathbb{Z} \).