1. Compute:
\[ 3^{1998} \mod 4 \quad 4^{26} \mod 14 \quad 2^{100} \mod 13. \]

Recall: [Fermat's Little Theorem]: If \( p \) prime and \( a \) not multiple of \( p \), then \( a^{p-1} \equiv 1 \mod p \).

Recall: If \( r \mod n \equiv a \) and \( s \mod n \equiv b \), then \( a + b \equiv rs \mod n \) and \( a \cdot b \equiv r \cdot s \mod n \).

\[
\begin{align*}
3^{1998} & \equiv 3 \mod 4 \\
4^{26} & \equiv 4 \mod 14 \\
2^{100} & \equiv 2 \mod 13
\end{align*}
\]

\[
\begin{align*}
4^1 & \equiv 4 \\
4^2 & \equiv 2 \\
4^3 & \equiv 8 \\
4^4 & \equiv 8 \\
4^5 & \equiv 2 \\
4^6 & \equiv 8 \\
4^7 & \equiv 2 \\
\end{align*}
\]

\[
\begin{align*}
3^1 & \equiv 3 \\
3^2 & \equiv 9 \\
3^3 & \equiv 3 \\
\end{align*}
\]

So, \( 4^k \mod 14 \equiv 4 \) if \( k \equiv 1 \mod 3 \)

\( 4^5 \mod 14 \equiv 16 \equiv 2 \mod 14 \)

\( 4^6 \mod 14 \equiv 8 \equiv 0 \mod 3 \).

\[
\begin{align*}
4^{26} \mod 14 & \equiv 2 \\
3^{1998} \mod 4 & \equiv 3 \\
2^{100} \mod 13 & \equiv 2
\end{align*}
\]

2. Compute: \( 2^{100} \mod 13. \)

Here, \( p = 13 \) prime and \( a = 2 \) not a multiple of \( 13 \). So, by Fermat's Little Theorem:

\[
2^{12} \equiv 1 \mod 13.
\]

\[
\begin{align*}
2^{100} & \equiv (2^{12})^{8} \cdot 2^{4} \mod 13 \\
& \equiv 1 \cdot 2 \mod 13 \\
& \equiv 2 \mod 13.
\end{align*}
\]

\[
\begin{align*}
2^{100} & \equiv 2 \mod 13 \\
3^{1998} & \equiv 3 \mod 13 \\
4^{26} & \equiv 2 \mod 13
\end{align*}
\]
2. Show that \( n^2 - 1 \) is always divisible by 3 provided that \( n \) itself is an integer not divisible by 3.

WTS \( n^2 - 1 \equiv 0 \mod 3 \), if \( n \) is not divisible by 3, then either \( n \equiv 1 \mod 3 \) or \( n \equiv 2 \mod 3 \).

**Case I:** \( n \equiv 1 \mod 3 \Rightarrow n^2 - 1 \mod 3 \equiv (1^2 - 1) \mod 3 \equiv 0 \mod 3 \).

**Case II:** \( n \equiv 2 \mod 3 \Rightarrow n^2 - 1 \mod 3 \equiv 2^2 - 1 \mod 3 \equiv 3 \mod 3 \equiv 0 \mod 3 \).

3. Given that \( n \equiv 2 \mod 7 \), is \( 5n^3 - a^2 \equiv 5 \cdot 2^3 - 2^2 \mod 7 \) ?

\[
5 \cdot 2^3 - 2^2 \mod 7 \equiv 36 \mod 7 \equiv 1 \mod 7.
\]

Consider \( n = 9 \), \( 9 \equiv 2 \mod 7 \).

\[
5 \cdot 9^3 - 2^2 \mod 7 = 5 \cdot (9)^3 - 2^9 \mod 7 \equiv 5 \cdot 2^3 - 2^9 \mod 7
\]

\[
eq 5 - 1 \mod 7 = 4 \mod 7.
\]

So, when \( n = 9 \), \( n \equiv 2 \mod 7 \), but \( 5 \cdot 9^3 - 2^9 \equiv 4 \mod 7 \) \( \neq 1 \mod 7 \). Thus, this is not true.

13. Prove that \( n^3 + 5n \) is divisible by 6 for every real.

WTS \( n^3 + 5n \equiv 0 \mod 6 \).

It suffices to show that \( n^3 + 5n \) is divisible by both 2 and 3 (i.e., \( n^3 + 5n \equiv 0 \mod 2 \) and \( n^3 + 5n \equiv 0 \mod 3 \)).
Either $n \equiv 0 \mod 2$ or $n \equiv 1 \mod 2$.

If $n \equiv 0 \mod 2$, then $n^2 + 5n \equiv 0^2 + 5(0) \mod 2 \equiv 0 \mod 2$.

If $n \equiv 1 \mod 2$, then $n^2 + 5n \equiv 1^2 + 5(1) \mod 2 \equiv 6 \mod 2 \equiv 0 \mod 2$.

Either $n \equiv 0 \mod 3$, $n \equiv 1 \mod 3$, or $n \equiv 2 \mod 3$.

If $n \equiv 0 \mod 3$, then $n^2 + 5n \equiv 0 \mod 3$.

If $n \equiv 1 \mod 3$, then $n^2 + 5n \equiv 1^2 + 5(1) \mod 3 \equiv 0 \mod 3$.

If $n \equiv 2 \mod 3$, then $n^2 + 5n \equiv 2^2 + 5(2) \mod 3 \equiv 8 + 10 \mod 3 \equiv 0 \mod 3$.

Therefore, $n^2 + 5n$ is divisible by $2 + 3 \equiv 0 \mod 3$, and hence is divisible by 6.

14. Show that the last digit of a nonnegative integer $n$ is its remainder when divided by 10. Then determine the last digit in the decimal expansion of $4^{100}$.

Suppose $n$ has $K$ digits, i.e., $n = n_1n_2\ldots n_K$.

$= n_1 \cdot 10^1 + n_2 \cdot 10^2 + \ldots + n_{K-1} \cdot 10^{K-1} + n_K$.

(E.g., $7392 = 7 \cdot 1000 + 3 \cdot 100 + 9 \cdot 10 + 2 = 7 \cdot 10^3 + 3 \cdot 10^2 + 9 \cdot 10 + 2$)

Since $10^i \equiv 0 \mod 10$ for $1 \leq i \leq K-1$ we have that $n \equiv n_K \mod 10$ (i.e., the remainder of $10^i n$ is the last digit $n_K$).

So, to find the last digit of $4^{100}$ it suffices to compute $4^{100} \mod 10$.

$4^1 \equiv 4 \mod 10$

$4^2 \equiv 16 \equiv 6 \mod 10$

$4^3 \equiv 24 \equiv 4 \mod 10$

$4^4 \equiv 16 \equiv 6 \mod 10$

$\ldots$ etc.
So, \( 4^k \mod 10 = 4 \) if \( k \) odd
\[ = 6 \] if \( k \) even.

\[ : 4^{100} \equiv 6 \mod 10 \implies \text{last digit of } 4^{100} \text{ is 6}. \]