Background: In each of the following, what is the independent variable and which is the dependent variable.

(1) \( y(x) = x^3 \). \( x \) independent, \( y \) dependent.

(2) \( \frac{dx}{dt} = 2 \). \( x \) dependent, \( t \) independent.

(3) \( \frac{dy}{dx^2} + \frac{dy}{dx} = y + x \). \( x \) and \( t \) independent, \( y \) dependent.

Notation: \( \frac{dy}{dx} = y_x = y' \).

Note: Dependent variables should be thought of as functions. e.g.: \( y(x) = x^2 \) is sometimes written: \( y = x^2 \) when it is clear \( y \) is dependent. i.e., \( y : \mathbb{R} \rightarrow \mathbb{R} \) \( x \rightarrow x^2 \).

Definition: An equation containing the derivatives of one or more unknown functions (dependent variables) w.r.t. a single independent variable is called an ordinary differential equation (ODE).

e.g.: \( \frac{dy}{dx} = x^3 \) is an ODE. Its solution is:

\[
y = \int x^3 \, dx = \frac{1}{4} x^4 + c. \text{ (on } (-\infty, \infty) \text{)}.\]
\[ e.g. \quad y'' + x^8 y''' + y'' = 0, \quad b \quad \frac{dy}{dx} + 7y = \left( \frac{dy}{dx} \right)^5 = 8, \]

\[ C \quad \frac{dt}{dx} + \frac{d^2y}{dt^2} = t^2 + xy \quad \text{are all examples of ODE's.} \]

\[ \text{Def}^n: \quad \text{An eq'n involving partial derivatives of one or more unknown functions of two or more independent variables is called a partial differential eq'n (PDE).} \]

\[ e.g. \quad \frac{d^2y}{dx^2} + \frac{dy}{dt} = 8 \quad \text{is a PDE. (It has a independent variables } x \text{ and } t).} \]

\[ * \quad \text{In this course, we will only consider ODE's. Therefore, instead of writing ODE's, I'll write DE.} * \]

\[ \text{Def}^n: \quad \text{The order of a DE is the order of the highest derivative in the eq'n.} \]

\[ e.g.: \quad a \quad \text{third-order}, \quad b \quad \text{and-order}, \quad c \quad \text{4th-order.} \]

\[ \text{Notation:} \quad \text{An } n^{th} \text{-order DE in one dependent variable is often expressed as } F(x, y, y', \ldots, y^{(n)}) = 0, \text{ where } F: \mathbb{R}^{n+2} \rightarrow \mathbb{R}. \]

\[ \text{Def}^n: \quad \text{An } n^{th} \text{-order DE } F(x, y, y', \ldots, y^{(n)}) = 0 \text{ is linear if } F \text{ is linear in the variables } y, y', \ldots, y^{(n)}. \]

\[ \text{I.e. } \quad F(x, y, y', \ldots, y^{(n)}) = a_0(x) \frac{dy}{dx} + a_1(x) y + a_2(x) y' + \ldots + a_n(x) y^{(n)} = 0. \]

\[ \text{Def}^n: \quad \text{A DE is nonlinear is just a DE that is not linear.} \]
Example 1: \( y^3 + \frac{dy}{dx}^3 + xy + z = 0. \) \( \triangleleft \) **Linear**

Example 2: \( x^2 y + \left( \frac{dy}{dx} \right)^3 = 0. \) \( \triangleleft \) **Nonlinear, b/c \( \frac{dy}{dx}^3 \)**

Example 3: \( \sin y + \frac{dy}{dx} = 0. \) \( \triangleleft \) **Nonlinear, b/c \( \sin y \)**

Example 4: \( y \frac{dy}{dx} + x = 0. \) \( \triangleleft \) **Nonlinear, b/c \( y \frac{dy}{dx} \)**

**Definition:** A function \( y: \mathbb{R} \rightarrow \mathbb{R} \) is called \( C^n \) on an interval if it possesses at least \( n \) derivatives that are continuous on \( I \).

**Example 1:** Any polynomial is \( C^n \), since polynomials are continuous and the derivative of a polynomial is a polynomial.

- \( y = \frac{1}{x} \) is continuous at all points except zero.
- \( \Rightarrow \) it's \( C^0 \) on \( (0, \infty) \) and \( (-\infty, 0) \).
- Its \( n \)th derivative is \( \frac{c}{x^n} \) for some constant \( c \).
- \( \frac{1}{x} \) is \( C^n \) on any interval not containing zero.

**Definition:** A solution of an \( n \)th-order DE on an interval \( I \) is any function \( y, C^n \) on \( I \), which, when substituted into the DE reduces the equation to an identity.

- \( \text{i.e., } F(x, y, y', \ldots, y^{(n)}) = 0 \text{ for all } x \in I \).

**Example 2:** Consider the DE \( y' = y \cos x \).

- \( y = C e^{\sin x} \) is an explicit solution on \( (-\infty, \infty) \), since \( y = \cos x \cdot e^{\sin x} \) is continuous on \( (-\infty, \infty) \) and \( y \cos x = C e^{\sin x} \cos x = y \).
Notice: A solution includes 2 things: a function $f$ on an interval $I$.

**Example:** The graph of $y = \frac{1}{x}$ looks like:

The DE $xy' + y = 0$ has $y = \frac{1}{x}$ as a solution on $(0, \infty)$ since $y' = -\frac{1}{x^2}$ cont. on $(0, \infty)$ & $xy' + y = x(-\frac{1}{x}) + \frac{1}{x} = 0$.

The graph of the solution is:

**Definition:** An implicit solution of a DE on an interval $I$ is a relation $G(x, y) = 0$ s.t. $I$ at least one function $y$ that satisfies both the DE & $G(x, y) = 0$.

**Example:** Consider the DE $\frac{dy}{dx} = -\frac{x}{y}$.

$x^2 + y^2 = 25$ is an implicit solution on $(0, 5)$, since $y = \sqrt{25 - x^2}$ satisfies $x^2 + y^2 = 25$ & the DE:

$\frac{dy}{dx} = \frac{-2x}{2\sqrt{25 - x^2}} = -\frac{x}{\sqrt{25 - x^2}}$. Here, $y = \sqrt{25 - x^2}$ is an explicit solution on $(0, 5)$.

**Definition:** A solution of a DE containing one arbitrary constant $c$ is called a one-parameter family of solutions. A solution with no constants is an $n$-parameter family. A solution with no arbitrary constants is a particular solution.
The DE \( \frac{dy}{dx} = 2x \) has a one-parameter family of solutions \( y = x^2 + C \). \( y = x^2 + 1 \) would be a particular solution. (Both solutions on \((-\infty, \infty)\).

**Def.** A singular solution of a DE is a solution which is not a member of a family of solutions of the DE.

Consider the DE \( \frac{dy}{dx} = xy^2 \). \( y = (\frac{4}{3}x^3 + C)^{1/2} \) is a family of solutions of the DE on \((-\infty, 0)\).

Since \( y = C \) and \( \frac{dy}{dx} = 2(x^2) \cdot 2C \), then \( C = xy^2 \).

But \( y = 0 \) is also a solution, since \( \frac{dy}{dx} = 0 = x(0)^{1/2} \).

However, there's no value of \( C \) which makes \( y = (\frac{4}{3}x^3 + C)^{1/2} \) equal to zero. \( \therefore 0 \) is a singular solution.

**Def.** A system of DEs is a system of 2 or more eqns involving derivatives of 2 or more unknown functions of a single independent variable.

A solution of such a system of \( N \) eqns are \( N \) functions \( y_1(x), \ldots, y_N(x) \) on a common interval \( I \), that satisfy the system on this interval.

**e.g.** \( \begin{cases} \frac{dx}{dt} = x + 3y \\ \frac{dy}{dt} = 5x + 3y \end{cases} \) is a system of 2 DEs with solution \( \begin{cases} x = 3e^{-2t} + 3e^{6t} \\ y = -e^{-2t} + 5e^{6t} \end{cases} \) \( \text{defined on } (-\infty, \infty) \).

Indeed, \( \frac{dx}{dt} = -2e^{-2t} + 18e^{6t} = x + 3y \).

\( \frac{dy}{dt} = 2e^{-2t} + 30e^{6t} = 5x + 3y \).
1.2: IVP: An $n^\text{th}$-order initial value problem (IVP) is the problem of solving an $n^\text{th}$ order DE $y^{(n)} = F(x, y, y', ..., y^{(n-1)})$ on some interval $I$ containing $x_0$, subject to the conditions $y(x_0) = y_0$, $y'(x_0) = y_1$, ..., $y^{(n-1)}(x_0) = y_{n-1}$ for some constants $y_i \in \mathbb{R}$.

**Example 1:** Solve the IVP \[ \begin{cases} y' = y \cos x \\ y(0) = a. \end{cases} \]

- $y = ce^{\sin x}$ on $(-\infty, \infty)$ solves $\text{DE}$ since $y' = c \cos x \cdot e^{\sin x}$.
- $y(0) = a \Rightarrow a = ce^0 \Rightarrow c = a$.

\[ \therefore \text{solution to IVP is} \ y = ae^{\sin x} \ on \ (-\infty, \infty). \]

**Example 2:** One could show $y = c_1 e^x + c_2 e^{-x}$ is a 2-parameter family of solutions of the IVP $y'' - y = 0$. Find a solution to the IVP $y'' - y = 0$, $y(0) = 1$, $y'(0) = 2$.

- $y' = c_1 e^x - c_2 e^{-x}$.
- $y(0) = 1 \Rightarrow 1 = c_1 + c_2$.
- $y'(0) = 2 \Rightarrow 2 = c_1 - c_2$.

\[ \therefore y = 3/2 e^x - 1/2 e^{-x} \ is \ a \ solution \ on \ (-\infty, \infty). \]

**Example 3:** Suppose the graph below is the graph of a function $y(x)$. Suppose also that $y(x)$ satisfies the DE $y' = f(x, y)$. Give an interval where the solution of the IVP $y' = f(x, y)$, $y(0) = -1$ is defined.

- Give 3 intervals where the solution of the DE $y' = f(x, y)$ is defined.
a) The interval must contain \((0, -1)\) or \((-1, 1)\) so any smaller interval in \((-1, 1)\) would do the job too.

b) We could choose \((-\infty, -1)\) or \((-1, 1)\) or \((1, \infty)\) or any smaller intervals in those 3.

Q) When does a solution to a first-order IVP exist \& when is such a solution unique?

**Theorem 1.3.1: Existence of a Unique Solution:**

Let \( R = [a,b] \times [c,d] \) contain the point \((x_0, y_0)\) in its interior. If \( f(x,y) \) 
and \( \frac{dy}{dx} \) are continuous on \( R \) \& \( J \) some interval \( I_0 = (y_0 - h, y_0 + h) \) in \( J \)
and contained in \([a,b] \times [c,d]\), then a unique function \( y(x) \) defined on \( I_0 \) \& \( y(x) \) is a solution of the IVP \( y' = f(x,y), \ y(x_0) = y_0 \).

Example: Consider the DE \( y' = 2y/x \). The function \( y(x) = cx^2 \) satisfies this DE. \( y' = 2cx = 2y/x \).

The graph of this 1-param. family looks like:
a. Does the IVP \( y' = 2y/x, \ y(0) = 0 \) have a unique solution?

b. Does the IVP \( y' = 2y/x, \ y(x) = y_0 \) for \( x_0 \neq 0 \) have a unique solution?

In the notation of Theorem 1.2.1, here

\[ f(x,y) = \frac{2y}{x}, \quad \frac{\partial f}{\partial y} = \frac{2}{x}. \]

These functions are cont. on \( \mathbb{R} \setminus \{0\} \times \mathbb{R} \) not containing 0.

\[ \therefore \text{By Theorem 1.2.1, if } y(x_0) = y_0 \text{ for } x_0 \neq 0, \text{ \exists some interval } I_0: \{x_0 - h, x_0 + h\} \text{ s.t. the IVP in } \quad \] 

\[ \text{has a unique solution.} \]

\[ \text{Since } x = 0 \text{ fails Theorem 1.2.1, this Theorem tells us nothing in } \quad \] 

\[ \text{However, looking at our family of graphs suggests there are infinitely many solutions to the IVP in } \quad \] 

\[ \text{Indeed, } y_1(x) = x^2 \text{ and } y_2(x) = 2x^2 \text{ are both s.t. } y(0) = 0 \text{ satisfy the IVP, } \quad \] 

\[ \text{in } \quad \] 

\[ \text{has no unique solution.} \]
25. Verify that the piece-wise-defined function
\[ y = \begin{cases} 
-x^2, & x < 0 \\
-x^2 + x^2, & x \geq 0
\end{cases} \]
is a solution of the DE
\[ xy' - 2y = 0 \text{ on } (-\infty, \infty). \]

The DE is 1st-order, so must show \( y \) satisfies
DE is \( C^1 \) on \( (-\infty, \infty) \).

\[ x < 0: \quad \frac{dy}{dx} (-x^2) = -2x. \quad xy' - 2y = x(-2x) - 2(-x^2) = 2x^2 = 0. \]

\[ x \geq 0: \quad \frac{dy}{dx} (x^2) = 2x. \quad xy' - 2y = x(2x) - 2(x^2) = 0. \]

Need to check if \( y' \) continuous on \( (-\infty, \infty) \).

\[ y' = \begin{cases} 
-2x, & x < 0 \\
2x, & x \geq 0
\end{cases} \]

\[ \lim_{x \to 10^-} y' = \lim_{x \to 10^+} y' = 0. \]

\[ \lim_{x \to 10^-} y = \lim_{x \to 10^+} y = 0. \]

\[ \therefore \text{ DE is } C^1 \text{ on } (-\infty, \infty). \]

26. In Ex. 5 we saw \( y = \sqrt[3]{5-x^2} \) and \( y = -\sqrt[3]{5-x^2} \)
are solutions of \( y' = -\frac{y}{x} \) on \((-5, 5)\). Explain why
\[ y = \begin{cases} 
\sqrt[3]{5-x^2}, & 0 \leq x < 5 \text{ Not a solution} \\
-x^2, & -5 \leq x \leq 0
\end{cases} \]

**We can see \( y \) is not**

\[ \lim_{x \to 0^+} y = 5. \]
\[ \lim_{x \to 0^-} y = -5. \]
\[ \lim_{x \to 10^+} y = -5. \]
\[ \lim_{x \to 10^-} y = 5. \]
Recall:

Differentiable at \( x \neq 1 \)

\( \text{cont. } e^{-x} \)

\( \Rightarrow \text{not cont. } e^{-x} \)

\( \Rightarrow \text{not diff. at } x \neq 1 \)

\( \Rightarrow y \text{ not cont. at } x = 5 \Rightarrow y \text{ not diff. at } x = 0 \)

\( \Rightarrow y \text{ not defined at } x = 0 \).

\[ \therefore y \text{ is not a solution on } (-5, 5). \]

\[ y \text{ not } C^1 \text{ on this interval.} \]

43. Given that \( y = \sin x \) is an explicit solution of the 1st-order DE \( y' = \sqrt{1 - y^2} \), find an interval of definition.

We need to find an interval where \( y = \cos x \) is defined everywhere, \( \Rightarrow y' = \cos x \text{ cont.} \).

\[ y' = \sqrt{1 - y^2} = \sqrt{1 - \sin^2 x} = \sqrt{\cos^2 x} = \cos x \]

\[ y = \sin x \Rightarrow y' = \cos x. \text{ So, } \cos x = \left| \cos x \right| \text{ when } \cos x > 0. \]

A possible interval would be \( (0, \pi) \).

The given graph represents the graph of an implicit solution \( G(x, y) = 0 \) of a DE \( y' = f(x, y) \), where \( G(x, y) = 0 \) implicitly defines several solutions of the DE. Mark off segments of the corr. to graphs of solutions. Estimate an interval of definition of each solution.

We need each \( f \) to be a function and differentiable.

\( f(x) = y ... \text{ can't have state of } \infty \)

Each \( x \) maps to one \( y \) value.