6.2. Solutions About Ordinary Points

In this section, we'll find solutions to 2nd-order linear DEs with nonconstant coefficients.

**Def:** A point \( x = x_0 \) is said to be an ordinary point of \( y'' + P(x)y' + Q(x)y = 0 \) if both \( P(x) \) and \( Q(x) \) are analytic at \( x_0 \). A point that is not an ordinary point is said to be a singular point of the DE.

**E.g. 7** \( y'' + e^x y' + (\cos x) y = 0 \).

\( e^x \) and \( \cos x \) are analytic for all values of \( x \) (i.e., they can be represented by their Taylor series at any point \( x_0 \)). \( \therefore \) Every \( x \) is an ordinary point.

**E.g. 7** \( az^2 y'' + a_1 y' + a_0 y = 0 \). Homogeneous DE w/ constant coef. \( a, z \).

If \( a_2 \neq 0 \), then \( P(x) = \frac{a_1}{a_2} + Q(x) = \frac{a_0}{a_2} \). Constant functions are analytic for all \( x \). \( \Rightarrow \) Every \( x \) is an ordinary point.

**E.g. 7** \( y'' + 7y' + 7y = 0 \). Find an ordinary & singular point.

Here \( P(x) = \frac{7}{7} + Q(x) = 7 \).

Recall: Rational Functions are analytic everywhere except where the denominator is zero. \( \therefore \) \( P(x) \) not analytic at \( x = 1 \), \( \Rightarrow \) \( x = 1 \) singular point. All other points are ordinary points.

**E.g. 7** \( y'' + \frac{1}{x+y} + 7y = 0 \). \( x = \pm i \) singular point.

\( \ast \) We can consider Complex Numbers too!
Theorem 6.2.1: [Existence of Power Series Solutions]

If \( x = x_0 \) is an ordinary point of \( a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_0(x)y = 0 \), we can always find a linearly independent solutions in the form of a power series centered at \( x_0 \), i.e.,

\[
y = \sum_{n=0}^{\infty} c_n (x-x_0)^n.
\]

A power series solution converges at least on some interval defined by \( |x-x_0| < R \), where \( R \) is the distance from \( x_0 \) to the closest singular point.

This distance \( R \) is a lower bound for the radius of convergence. The solution \( y \) is called a solution about the ordinary point \( x_0 \).

**E.g. 7**

Find the minimum radius of convergence of a power series solution of \( (x^2 - 2x + 10)y'' + xy' - 4y = 0 \).  

a) about the ordinary point 0.  
b) about the ordinary point 1.

Here \( P(x) = \frac{x}{x^2 - 2x + 10} \) and \( Q(x) = \frac{-4}{x^2 - 2x + 10} \).

\[
x = 2 \pm \sqrt{4 - 4(10)} = 1 \pm 3i. \quad \text{So, this eqn has singular points 1+3i and 1-3i.}
\]

**Recall:** The distance b/w two complex numbers \( a + bi \) and \( a_2 + b_2i \) is given by \( \sqrt{(a_1 - a_2)^2 + (b_1 - b_2)^2} \).

\( a = 0 + 0i; \) so the distance b/w 0 and the closest singular point is \( \sqrt{3^2 + 1^2} = \sqrt{10}. \)  

\( \therefore \) min. rad. of conv. \( R = \sqrt{10}. \)
The distance $b/w$ $1$ and $1 \pm 3$ is: $\sqrt{(1-1)^2 + 3^2} = 3$. 

$R = 3$ is the min. rad. of convergence.

**Method of Solution**

[Finding a Power Series Solution]

[Of a Homogeneous Linear 2nd Order DE]:

For the sake of simplicity, we'll only find power series solutions about the ordinary point $x = 0$.

1. **Substitute** $y = \sum_{n=0}^{\infty} c_n x^n$ into the DE. 
   [This is our guess for the solution, but we want to locate the $c_n$s. You'll have to compute $y', y''$ here].

2. **Combine the Series.** [i.e., in each, make a substitution for $x = x_0$, and add the series together].

3. **Equate all coefficients to the RHS of the equation to determine the coefficients $c_n$.** [we use the Identity Property here].

4. This leads us to two distinct sets of coefficients, so that we have a distinct power series $y_1$ and $y_2$. Therefore, the general solution is $y = c_0 y_1 + c_1 y_2$.

**Example:** Find a general solution to $2y'' + xy' + y = 0$.

Here $p(x) = \frac{x}{2} + q(x) = \frac{1}{2}$, so all $x$'s are ordinary points.

In particular, $x = 0$ is an ordinary point. By Theorem 6.3.1 we're guaranteed a linearly independent solution in the form of a power series centered at $0$. 

\[ y = \sum_{n=0}^{\infty} c_n x^n, \quad y' = \sum_{n=1}^{\infty} n c_n x^{n-1}, \quad y'' = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}, \]
\[ a y'' + x y' + y = \sum_{n=0}^{\infty} c_n n(n-1) x^n + \sum_{n=0}^{\infty} c_n n x^n + \sum_{n=0}^{\infty} c_n x^n. \]
\[ \Rightarrow \quad \sum_{n=0}^{\infty} 2 c_n n (n-1) x^n = \sum_{n=0}^{\infty} c_k x^k, \quad k = n - a \]
\[ \Rightarrow \quad a c_{k+a} (k+a)(k+1) x^k = \sum_{k=0}^{\infty} c_k x^k \]
\[ \Rightarrow \quad 4 c_2 + c_0 + \sum_{k=1}^{\infty} \left[ 2 a c_{k+2} (k+a)(k+1) + (k+1) c_k \right] x^k = 0. \]
\[ \Rightarrow \quad 4 c_2 + c_0 = 0 \quad \text{and} \quad 2 a c_{k+2} (k+a)(k+1) + (k+1) c_k = 0. \]
\[ \Rightarrow \quad c_{k+a} = \frac{-c_k}{2(k+a)}. \]

So, we have:
\[ C_0, C_1, C_2 = -\frac{1}{4} C_0, \quad C_3 = \frac{-c_1}{2.3}, \quad C_4 = \frac{-c_2}{2.4}, \quad C_5 = \frac{-c_3}{2.5}, \quad C_6 = \frac{-c_4}{2.8}, \quad C_7 = \frac{-c_5}{3.7}, \quad C_8 = \frac{-c_6}{2.9}. \]
We can see the pattern for the coefficients is:
\[ C_{an} = (-1)^n c_0 \quad \text{and} \quad C_{an+1} = (-1)^{n+1} c_1 \]
\[ = \frac{1}{a^n n!} \quad \text{and} \quad \frac{(-1)^{n+1} c_1}{a^{n+1} (n+1)!} \]

Two linearly independent solutions are:
\[ y_1 = \frac{1}{2^n n!} \sum_{n=0}^{\infty} (-1)^n x^n \quad \text{taking} \quad c_0 = 1 \quad \text{and} \quad c_1 = 0 \]
\[ y_2 = \frac{1}{2^{n+1} (n+1)!} \sum_{n=0}^{\infty} (-1)^n x^{n+1} \quad \text{taking} \quad c_0 = 0 \quad \text{and} \quad c_1 = 1 \]

A general solution is given by \( y = ay_1 + by_2 \).
Since our DE has no singular points, this solution converges on \((-\infty, \infty)\) [by Theorem 6.2.1].

Note: Sometimes it may be too difficult to identify a formula for the recurrence relation. In this case, you'll be asked to write down the first 3 or 4 terms in the solution.

Non-polynomial coefficients: If the coefficients in our 2nd-order linear DE are not polynomials, then first write the coefficient in terms of its power series [we can do this since \( p(x) \) and \( q(x) \) are both analytic], then solve as normal.

Non-homogeneous linear DEs: A point \( x_0 \) is an ordinary point of a non-homogeneous linear DE \( y'' + p(x)y' + q(x)y = f(x) \) if \( p(x), q(x), \) and \( f(x) \) are analytic at \( x_0 \). Theorem 6.2.1.
IF we just want the first few terms of \( y_1 \) and \( y_2 \), obtain \( y_1 \) by setting \( c_0 = 1 \) and \( c_1 = 0 \), and \( y_2 \) by setting \( c_0 = 0 \) and \( c_1 = 1 \).

extends to these DE's. We solve them in the exact same way as the homogeneous case [just bring \( F(x) \) to the other side: \( y'' + p(x)y' + q(x)y = F(x) = 0 \), or at the end just equate coefficients.

\[ y = \sum_{n=0}^{\infty} c_n x^n, \quad y' = \sum_{n=1}^{\infty} n c_n x^{n-1}, \quad y'' = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}. \]

\[ y'' - xy' + 2y = \cos x \]

\[ c_n \left( n(n-1) x^{n-2} - \sum_{k=1}^{n} \frac{c_k}{c_k} x^{n-k} + 2 \sum_{k=0}^{n-2} \frac{c_k}{c_k} x^{n-k} \right) = \sum_{k=0}^{\infty} c_k x^k \]

\[ 2c_2 + 2c_0 + \sum_{k=1}^{\infty} \left[ c_{k+2} (k+2)(k+1) + k c_k + 2 c_k \right] x^k = \frac{1}{1-x^2} + \sum_{k=0}^{\infty} \left[ c_{k+2} (k+2)(k+1) + k c_k + 2 c_k \right] x^k \]

\[ 2c_2 + 2c_0 = 1 \quad \text{and} \quad 6c_3 + c_1 = 0. \]

\[ c_2 = \frac{1}{4} \quad \text{and} \quad c_3 = \frac{1}{6} c_1 \]

\[ y = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + \left( \frac{1}{2} - c_0 \right) x^2 + \left( \frac{1}{6} c_1 \right) x^3 + \ldots. \]

\[ y = \left[ \frac{1}{2} x^2 + \ldots \right] + c_0 \left[ 1 - x^2 + \ldots \right] + c_1 \left[ x - \frac{1}{6} x^3 + \ldots \right], \quad \text{and} \]

\[ y_1 = 1 - x^2 + \ldots, \quad y_2 = x - \frac{1}{6} x^3 + \ldots, \quad y = y_1 + y_2. \]
**Example 7:** Consider \( y'' + e^x y' - y = 0. \) Find 2 power series solutions.

**First 3 terms of each:**

\[
y = \sum_{n=0}^{\infty} c_n x^n, \quad y' = \sum_{n=1}^{\infty} cn x^{n-1}, \quad y'' = \sum_{n=2}^{\infty} \frac{n(n-1)}{2} c_n x^{n-2}
\]

\[
e^x = 1 + x + \frac{x^2}{2!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!} x^n.
\]

So, \( y'' + e^x y' - y = \sum_{n=2}^{\infty} \frac{n(n-1)}{2} c_n x^{n-2} \left[ \sum_{n=0}^{\infty} \frac{x^n}{n!} \right] \]

\[
= \sum_{n=2}^{\infty} \frac{n(n-1)}{2} c_n x^{n-2} \left[ \sum_{n=0}^{\infty} \frac{x^n}{n!} \right] = \left[ \sum_{n=2}^{\infty} \frac{n(n-1)}{2} c_n x^{n-2} \right] \cdot \left[ \sum_{n=0}^{\infty} \frac{x^n}{n!} \right]
\]

\[
= \left[ \sum_{n=2}^{\infty} \frac{n(n-1)}{2} c_n x^{n-2} \right] = \left[ \sum_{n=0}^{\infty} \frac{x^n}{n!} \right] \text{series}
\]

**Since we just want 3 terms, let's work out each series.**

\[
= \left[ \sum_{n=2}^{\infty} \frac{n(n-1)}{2} c_n x^{n-2} \right] = \left[ \sum_{n=0}^{\infty} \frac{x^n}{n!} \right] \text{series}
\]

\[
= a_{c_2} - c_0 + c_1 = 0 \quad \Rightarrow \quad 6c_3 - c_1 + (ac_2 + c_1) = 0
\]

\[
= 12c_2 - c_0 + c_1 = 0 \quad \Rightarrow \quad 6c_3 - c_1 + (ac_2 + c_1) = 0
\]

\[
= 2c_2 = \frac{1}{3}c_0 - \frac{1}{3}c_1 \quad \Rightarrow \quad 6c_3 - c_1 + (ac_2 + c_1) = 0
\]

\[
= c_2 = \frac{1}{3}c_0 - \frac{1}{3}c_1 \quad \Rightarrow \quad 6c_3 - c_1 + (ac_2 + c_1) = 0
\]

\[
= c_3 = -\frac{1}{3}c_2 + \frac{1}{3}c_3 \quad \Rightarrow \quad 6c_3 - c_1 + (ac_2 + c_1) = 0
\]

\[
c_4 = -\frac{1}{12}c_2 + \frac{1}{12}c_3
\]

**When \( c_0 = 1, c_1 = 0: \)**

\[
c_2 = \frac{1}{3}, \quad c_3 = -\frac{1}{6}, \quad c_4 = \frac{1}{12} \left( \frac{1}{3} - \frac{1}{4} \right)
\]

\[
= \frac{1}{12}(-\frac{1}{2}) - \frac{1}{4} \cdot 2
\]

\[
= \frac{1}{12} \left( \frac{1}{2} - \frac{1}{4} \right) - \frac{1}{4}
\]

So, \( y_1 = 1 + \frac{1}{3} x^2 - \frac{1}{6} x^3 + \cdots. \)

**When \( c_0 = 0, c_1 = 1: \)**

\[
c_2 = -\frac{1}{3}, \quad c_3 = \frac{1}{6}, \quad c_4 = \frac{1}{12} \left( \frac{1}{2} - \frac{1}{4} \right)
\]

\[
= -\frac{1}{12} \left( \frac{1}{3} - \frac{1}{4} \right)
\]

\[
= -\frac{1}{12} \left( \frac{1}{12} \right)
\]

\[
= -\frac{1}{12}
\]

So, \( y_2 = x - \frac{1}{3} x^2 + \frac{1}{6} x^3 + \cdots. \)
What is the minimum radius of convergence for $y_1$ and $y_2$?

Here $p(x) = e^x + Q(x) = -1$, $e^x - 1$ have radius of convergence $(-\infty, \infty)$. The DE has no singular points, so the minimum radius of convergence is $R = \infty$.

E.g. 7: Solve the IVP $2y'' + xy' + y = 0$, $y(0) = 1$, $y'(0) = 0$.

From the 1st example we know we have the recurrence relations:

$$c_n = \frac{(-1)^n c_0}{2^n n!} + c_{n+1} = \frac{(-1)^n c_1}{2^n [1 \cdot 3 \cdot 5 \ldots (2n+1)]}$$

$y(0) = 1 \Rightarrow \sum_{n=0}^{\infty} c_n x^n = 1 \Rightarrow c_0 = 1$.

$y'(0) = 0 \Rightarrow \sum_{n=1}^{\infty} n c_n x^{n-1} = 0 \Rightarrow c_1 = 0 \Rightarrow$ all odd terms are zero.

$y = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n n!} x^n = e^{-\frac{x^2}{4}}$.

Since $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, $x^n \leq e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, $n! \leq e^{\frac{x^2}{4}}$

$\Rightarrow \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n n!} x^n = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!}$.