4.5: Undetermined Coefficients - Annihilator Approach

- In this section we'll learn how to solve certain types of nonhomogeneous DE's with constant coefficients:

\[ a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = g(x) \]

For the method in this section to work, we need \( g(x) \) to be a function consisting of finite sums of products of constants, polynomials, exponential functions \( e^{ax} \), sines, and cosines.

Recall: It's sometimes convenient to write \( L(y) = g(x) \) or \( a_0 y + \cdots + a_n y^{(n)} = g(x) \).

Here \( L \) is the linear operator \( L = a_0 D^n + \cdots + a_n D + a_0 \).

\[ 3y'' + 15y = x^2 \iff (3D^2 + 15)(y) = x^2 \]

Def: A linear operator \( L \) with constant coef. is said to be an annihilator of a sufficiently differentiable function \( f \) if \( L(f(x)) = 0 \).

\[ L = D^n \] annihilates \( f + x \) since \( \frac{d^2}{dx^2}(y) = \frac{d^2}{dx^2}(x) \).

\[ \star \text{Notice: The set of functions that are annihilated by } L = a_0 D^n + \cdots + a_n D + a_0 \text{ are those functions which are in the set of solutions of the homog. DE } L(y) = 0. \]

\[ \leq \text{They can be obtained from the general solution of } L(y) = 0. \]

\[ \star \text{Which functions are annihilated by } D^n? \]

To find these we want to consider \( D^n(y) = 0 \) (i.e., \( y^{(n)} = 0 \)). The
general solution is \( c_1 e^x + c_2 x e^x + \ldots + c_{n-1} x^{n-1} e^x \). 

**Example 2** Which functions are annihilated by \((D - \alpha)^n\)?

\((D - \alpha)^n(y) = 0\) has aux. \( e^{\alpha t} \) \((m - \alpha) = 0\) \(\Rightarrow\) \(\alpha\) root of multiplicity \(n\).

**Example 3** Which functions are annihilated by \([D^2 - 2\alpha D + (\alpha^2 + \beta^2)]^n\)?

\[ m^2 - 2\alpha m + (\alpha^2 + \beta^2) = 0 \]

\[ m = 2 \alpha \pm \sqrt{4\alpha^2 - 4\alpha^2 - 4\beta^2} = 2 \alpha \pm i\beta \]

**Example 4** Which functions are annihilated by \((D - \alpha)^n\)?

**Properties:**

1. If \( y_1 \) and \( y_2 \) are annihilated by \( L \), then \( L(y_1 + y_2) = 0 \).

2. Suppose \( L_1(y_1) = 0 \), \( L_2(y_2) = 0 \), but \( L_1(y_2 + y_1) \neq 0 + L_2(y_1) \neq 0 \).

Then \( L_1(y_1 + y_2) = 0 \).

**Example 5** Find an operator which annihilates \( e^{-x} \sin x - e^{-x} \cos x \).

We know \((D + 2D + 2a)\) annihilates \( e^{-x} \sin x \), \((D^2 - 4D + 5)\) annihilates \( e^{x} \cos x \), so \((D^2 + 2aD + a)\) \((D^2 - 4D + 5)\) will annihilate \( e^{-x} \sin x - e^{x} \cos x \).
Method of Undetermined Coefficients - Annihilator Approach:

This is a method for finding the general solution for
\( L(y) = g(x) \), where \( g(x) \) finite sums/products of
functions: \( \exp(ax), \sin(x), \cos(x) \).

1. Find the general solution \( y_h \) for homog. eq. \( L(y) = 0 \).
2. Find an operator \( L_1 \) which annihilates \( g(x) \) & operate
   on both sides of \( L_1(y) = g(x) \) by \( L_1 \), the lowest possible
   order operator that does the job.
3. Find general solution for homog. eq. \( L_1 y = 0 \).
4. Delete from solution in 3 the terms that also
   appear in \( y_h \) in 1.
5. Substitute \( y_p \) found in 4 into \( L(y) = g(x) \). Match
   coeff. & solve for unknown coeff. in \( y_p \).
6. The general solution is \( y = y_h + y_p \).

Example:

Find the general solution for the nonhomog. DE
\[ y'' + 3y' = 4x - 5. \]

1. \( y'' + 3y' = 0 \) has aux. \( m^2 + 3m = 0 \), \( m = 0 \) or \( m = -3 \).
   General solution is \( y_h = c_1 + c_2 e^{-3x} \).

2. \( L^2 \) annihilates \( 4x - 5 \) & has the lowest possible order.
   \( L = D^2 + 3D \), so \( L(L(y)) = D^2(D^2 + 3D) = D^4 + 3D^3 \).
So, we want to find general solution for \( y^{(4)} + 3y^{(3)} = 0 \).

Aux. eqn is \( m^4 + 3m^3 = 0 \) \( \Rightarrow \) \( m = 0 \) \( \oplus m = -3 \).

So, general solution is \( c_1 + c_2x + c_3x^2 + c_4 e^{-3x} \).

1. \( c_1 \) and \( c_4 e^{-3x} \) appear \( \Rightarrow \) 0 \( \oplus \) 2, so \( y_p = c_3x + c_4 x^2 \).

2. \( y_p' = c_3 + 2c_4x \). \( y_p'' = 2c_4 \). So, \( L(y_p) = g(x) \)

\( \Rightarrow 2c_4 + 3c_3 + 6c_4x = 4x - 5 \Rightarrow (2c_4)x + (2c_4 + 3c_3) = 4x - 5 \)

\( \Rightarrow 2c_4 + 3c_3 = -19 \) \( \Rightarrow c_3 = -19 \) \( \Rightarrow c_3 = -\frac{19}{9} \).

So, \( y_p = -\frac{19}{9} x + \frac{2}{3} x^2 \).

3. \( y_p'' = 2\frac{19}{9} - \frac{2}{3} x^2 \).

4. The general solution is \( y = c_1 + c_2 e^{-3x} + \frac{2}{3} x^2 - \frac{19}{9} \).

4.6: Variation of Parameters:

In section 4.5, we learned to solve \( L(y) = g(x) \) where \( g(x) \) was the sumproduct of polynomials \( e^x, \sin x, \cos x \) using undetermined coefficients. In this section, we'll solve \( L(y) = g(x) \) where there is no restrict on \( g(x) \).

Variation of Parameters can also be used to solve \( L(y) = g(x) \) where \( L \) doesn't have constant coefficients [see #23, 32 in 4.6 and see section 4.7].

Variation of Parameters For \( n=2 \) : \( a_2 y'' + a_1 y' + a_0 y = g(x) \)

\[ y_e = c_1 y_1 + c_2 y_2 \]

Find general solution for homog. \( a_2 y'' + a_1 y' + a_0 y = 0 \).

Compute Wronskian \( W(y_1(x), y_2(x)) \).
4. Put eq. in standard form $y'' + py' + qy = f(x)$.

5. Compute $W_1 = \begin{vmatrix} 0 & y_2 \\ f(x) & y_1' \end{vmatrix}$, $W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix}$.

6. Find $u_1 = \int \frac{W_1}{W} \, dx$ and $u_2 = \int \frac{W_2}{W} \, dx$.

A particular solution is $y_p = u_1y_1 + u_2y_2$ and the general solution is $y = y_c + y_p$.

Idea Behind Solution Method: Want to find a particular solution of the form $y = u_1y_1(x) + u_2y_2(x)$.

[Plug this into the original ODE, follow your nose and solve for $u_1$ and $u_2$].

Example: Find the general solution of $y'' + y = \tan x$.

4. $y'' + y = 0$ has aux. eq. $m^2 + 1 = 0 \Rightarrow m = \pm i$. $a = 0, b = 1$.

So $y_e = c_1 \cos x + c_2 \sin x$ is the general solution to homogeneous eqn.

2. $W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = (\cos x \cdot -\sin x - \cos x \cdot \sin x) = 0$.

Wronskian not zero, solutions linearly independent.

3. $y'' + y = \tan x$.

4. $W_1 = \begin{vmatrix} 0 & \tan x \\ \cos x & \sin x \end{vmatrix} = -\sin x \tan x$. $W_2 = \begin{vmatrix} \cos x & 0 \\ \sin x & \tan x \end{vmatrix} = \sin x \tan x$. 

Computes Wronskian $W(x)$.
u_1 = \int \frac{\sin^2 x}{\cos x} \, dx = -\int \frac{\sin^2 x}{\cos x} \, dx = -\int \frac{1}{\cos x} \cdot \cos x \, dx = -\ln |\sec x + \tan x| + C \cdot \sin x + C.

u_2 = \int \sin x \, dx = -\cos x + C.

So, \( y_p = \sin x \cos x - \ln |\sec x + \tan x| \cdot \cos x \). 
\( y = c_1 \cos x + c_2 \sin x - \ln |\sec x + \tan x| \cdot \cos x \) is a particular solution.

Variation of Parameters for General \( n \) case:

The method shown in the \( n = 2 \) case naturally generalizes.

\( y_c = c_1 y_1 + \ldots + c_n y_n \).

Here, \( u_k = \frac{W_k}{W} \), where \( W \) is the det. of the matrix obtained by replacing the \( k \)th column of the Wronskian by \( (0 \ 0 \ \ldots \ 0 \ f(x)) \) and \( W = W(y_1, \ldots, y_n) \).

\( y_p = u_1 y_1 + \ldots + u_n y_n \). General solution is \( y = y_c + y_p \).

4.7: Cauchy–Euler Eqn

In 4.3-4.6 we solved linear DE's wr constant coef.
In this section, we'll learn how to solve a special type of linear DE wr non-constant coef.

**Def**: A linear DE of the form \( a_n x^n y^{(n)} + a_{n-1} x^{n-1} y^{(n-1)} + \ldots + a_1 x y' + a_0 y = g(x) \)

is known as a Cauchy-Euler eqn

[Here the \( a_i \) are constants]
We solve these DE's in a manner very similar to sections 4.3 (Homog. w/ constant coeff.) and 4.6 (Variation of Parameters).

**Method of Solution:**

1. Solve the homog. eq'n \( a_n y^{(n)} + \ldots + a_1 y' + a_0 y = 0 \) by trying a solution of the form \( y = x^m \). In 4.3, we did this, but instead used \( y = e^{mx} \). Plug \( y = x^m \) into the eq'n. Each term \( a_k x^k y^{(k)} \) will become \( a_k x^m m(m-1)(m-2)\ldots(m-k+1) \).

2. We seek a solution on \( 0, \infty \). So factor \( x^m \) out of the above eq'n. \( y = x^m \) is a solution of the DE whenever \( m \) is a solution to the auxiliary eq'n.

3. Analogous to 4.3, if we have the following cases:

   a. **If** we have distinct roots \( m_1, \ldots, m_k \), the general solution will contain the linear combination \( c_1 x^{m_1} + \ldots + c_k x^{m_k} \).

   b. **If** we have a root \( m \) of multiplicity \( k \), then the general solution will contain the linear combination \( c_1 x^m + c_2 x^m \ln x + \ldots + c_k x^m (\ln x)^{k-1} \).

   c. **If** we have a complex conjugate pair \( \alpha \pm \beta i \), then the general solution will contain the linear combination \( x^\alpha [c_1 \cos(\beta \ln x) + c_2 \sin(\beta \ln x)] \).

4. **Use variation of parameters** to solve the nonhomog. Cauchy-Euler eq'n. Example obtained in steps 1-3.
e.g.7 Solve $2x^2y'' + 5xy' + y = x^2 - x$.

1. $y = x^m \Rightarrow 2x^m(m(m-1)) + 5x^m + x^m = 0$
   $\Rightarrow x^m \left(2m^2 + 3m + 1\right) = 0$
   $\Rightarrow (am+1)(m+1) = 0 \Rightarrow m = -1, \frac{-1}{a}$.

2. So, $y_c = c_1x^(-1) + c_2x^(-\frac{1}{2})$.

3. $W = \left| \begin{array}{cc} x^{-1} & x^{-\frac{1}{2}} \\ -x^{-2} & -\frac{1}{2}x^{-\frac{3}{2}} \end{array} \right| = -\frac{1}{2}x^{-\frac{1}{2}} + \frac{1}{2}x^{-\frac{3}{2}}$.

4. $W = \int \left[ \begin{array}{cc} -x^{-2} & x^{-1} \\ -x^{-2} & -\frac{1}{2}x^{-\frac{3}{2}} \end{array} \right] dx = -\frac{1}{3}x^3 + \frac{1}{2}x^2$.

5. $y = c_1x^{-1} + c_2x^{-\frac{1}{2}} - \frac{1}{3}x^2 + \frac{1}{2}x + \frac{2}{3}x^2 - \frac{2}{3}x$.

e.g.7 Solve $x^4y^{(4)} + 6x^3y^{(3)} + 9x^2y'' + 3xy' + y = 0$.

1. $y = x^m \Rightarrow x^m \left[ m(m-1)(m-2)(m-3) + 6m(m-1)(m-2) + 9m(m-1) + 3m+1 \right] = 0$

2. $(m^2-m)(m^2-5m+6) + 6(m^2-m)(m-2) + 9m^2 - 9m + 3m + 1 = 0$

3. $m^4 - 6m^3 + 11m^2 - 6m + 6m^3 - 18m^2 + 12m + 9m^2 - 6m + 1 = 0$

4. $m^4 + 2m^2 + 1 = 0 \Rightarrow (m^2 + 1)^2 = 0 \Rightarrow m = \pm i$. Double pair.
So, the general solution is:

\[ y = c_1 \cos(lhx) + c_2 \sin(lhx) + c_3 lhx \cos(lhx) + c_4 lhx \sin(lhx). \]