Math 2C03 - Class #6

Midterm Info:
- Wed. July 15th, 7pm - 8:15pm.
- HH104: Baswick: Thurs. - 05.
- Lecture in BSB105 from 8:30pm - 9:45pm.

Format:
- 3 Fill in the blank
- 6 Fill answer questions
- Approx. 85% WebWork type questions
- e.g. solve the following DE...[

Half of the battle is recognizing the type of DE:

1. Separable? Yes
   Linear? Yes
   Exact? Yes
   Solve

2.1, 2.3, 3.1

1st - order

It must be linear.
Constant coefficients?

3.4

Homogeneous?

4.7

Cauchy-Euler?

4.7

Must know a solution if already use variation of parameters too.

4.5

Anchored Approach

4.2

If non-homog. use variation of parameters

4.4

Non-linear

4.4

Reduction of order

Bernoulli?

Yes

Homogeneous?

Yes

F(Ax + By + c) = 0?

F(Ax + By + c) = 0?

Solve as in 2.3.

Can I find an integrating factor to make it exact?

Yes

solve corr. exact eq'n

Yes

make substitution to solve the corr. separable/ linear eq'n

No

must have made a mistake somewhere!

Yes

Can I find an integrating factor to make it exact?

No

solve corr. exact eq'n

Solve

Product of

Poly's, exponents, sines, cosines

No

Annihilating Approach

Yes

Variation of parameters

If non-homog. must have made a mistake somewhere!
Ch. 7: The Laplace Transform

Motivation: Laplace Transforms allow us to solve linear differential DE's with constant coefficients: \[ a_n y^{(n)} + \cdots + a_1 y' + a_0 y = g(x) \]. It's especially useful when \( g(x) \) has a jump discontinuity (advantage over techniques learned in Ch. 6). These types of DE's model phenomena in many real-world applications (e.g., electrical circuits, probability theory).

7.1: Definition of Laplace Transform:

Defn: Let \( F \) be a function defined for \( t \geq 0 \). Then the Laplace transform of \( F \) is the integral \( F(s) := \mathcal{L}(F(t)) = \int_0^\infty e^{-st} F(t) \, dt \), provided that the integral converges.

E.g. 1: Evaluate \( \mathcal{L}[f(t)] \).

\[ \mathcal{L}[f(t)] = \int_0^\infty e^{-st} f(t) \, dt = \lim_{b \to \infty} \int_0^b e^{-st} f(t) \, dt \]

\[ = \lim_{b \to \infty} \left[ -e^{-st} f(t) \right]_0^b + \lim_{b \to \infty} \int_0^b e^{-st} f'(t) \, dt \]

\[ = -e^{-st} f(b) + \lim_{b \to \infty} \int_0^b e^{-st} f'(t) \, dt \]

\[ = \left\{ \begin{array}{ll}
-\frac{e^{-st} f(t)}{s} & \text{if } s > 0 \\
\frac{1}{s} & \text{if } s = 0
\end{array} \right. \]

E.g. 2: Evaluate \( \mathcal{L}[f(t)] \).

\[ \mathcal{L}[f(t)] = \int_0^\infty e^{-st} f(t) \, dt = \frac{1}{s} \left[ 3 \int_0^\infty e^{-st} \, dt + \int_0^\infty e^{-st} \, dt \right] \]

\[ = \frac{1}{s} \left[ -e^{-st} \right]_0^\infty + \frac{1}{s} \left[ -e^{-st} \right]_0^\infty \]

\[ = \frac{1}{s} \left[ 0 - (-1) + 0 - (-1) \right] = \frac{2}{s} \]
Evaluate \( \int e^{5t^3} \, dt \).

\[
\int e^{5t^3} \, dt = \int e^{s} \, ds \quad \text{where} \quad s = 5t^3,
\]

so

\[
\int e^{s} \, ds = \frac{1}{5} e^{s} + C = \frac{1}{5} e^{5t^3} + C.
\]

We don't always write the restrictions on \( s \). It's understood that \( s \) is sufficiently restricted to guarantee convergence.

Transforms of some Basic Functions:

- \( a. \quad \mathcal{L}(t) = \frac{1}{s} \).
- \( b. \quad \mathcal{L}(t^n) = \frac{n!}{s^{n+1}} \quad \text{for} \quad n = 1, 2, 3, \ldots \).
- \( c. \quad \mathcal{L}(e^{-at}) = \frac{1}{s+a} \).
- \( d. \quad \mathcal{L}\left(\sin(kt)\right) = \frac{k}{s^2+k^2} \).
- \( e. \quad \mathcal{L}\left(\cos(kt)\right) = \frac{s}{s^2+k^2} \).
- \( f. \quad \mathcal{L}\left(\sinh(kt)\right) = \frac{k}{s^2-k^2} \).
- \( g. \quad \mathcal{L}\left(\cosh(kt)\right) = \frac{s}{s^2-k^2} \).

\[ \mathcal{L} \text{ is a linear transform:} \]

\[ \mathcal{L}(a f(t) + b g(t)) = a \mathcal{L}(f(t)) + b \mathcal{L}(g(t)). \]

This follows from the linearity of the integral.
A function \( F \) is piecewise continuous on an interval \([a,b]\) if \([a,b]\) can be subdivided into a finite number of subintervals \(t\), and in each subinterval \( t \), it is continuous and has a finite left and right limit. i.e., it has a finite number of breaks and doesn't blow up to infinity anywhere. (No vertical asymptotes, only jump discontinuities).

**Example:**

\[
\text{is piecewise cont. on } [a,b].
\]

\[
\text{not piecewise cont. on } [-1,0] \text{ b/c left and right limits approach } 0 \text{ at } -\infty.
\]

\( f(t) \) piecewise cont. on \([0,\infty)\) if piecewise cont. on \([0,b]\) for every \( b > 0 \).

**Defn:** A function \( F \) is of exponential order \( c \) if there exist constants \( c, M \geq 0, T > 0 \) s.t. \( |F(t)| \leq Me^{ct} \) for all \( t \geq T \).

**i.e.:** \[ \lim_{t \to \infty} \frac{|F(t)|}{e^{ct}} = L, \text{ where } L \geq 0 \text{ is finite.} \]

**Example:** If \( F \) is an increasing function, this says that the graph of \( F \) on \((T, \infty)\) does not grow faster than the graph of \( Me^{ct} \), where \( c \) is a positive constant.
Theorem 7.1.2: Existence of the Laplace Transform [sufficient conditions]

If \( F \) is piecewise continuous on \([0, \infty)\) and of exponential order \( c \), then \( \mathcal{L}\{F(t)\} \) exists for \( s > c \).

Proof: [see pg. 278]

7.2: Inverse Transforms & Transforms of Derivatives

Definition: Given a function \( F(s) \), if there is a function \( g(t) \) that is continuous on \([0, \infty)\) and satisfies \( \mathcal{L}\{g(t)\} = F(s) \), then \( g(t) \) is the inverse Laplace transform of \( F(s) \) and we write \( g(t) = \mathcal{L}^{-1}\{F(s)\} \).

Example: \( \mathcal{L}^{-1}\{\frac{1}{s+3}\} = e^{-3t} \), \( \mathcal{L}^{-1}\{\frac{3}{s+3}\} = 3e^{-3t} \), etc.

Example: Find \( \mathcal{L}^{-1}\{s^2 + 4s + 3\} \).

We know \( \mathcal{L}\{e^{-3t}\} = \frac{1}{s+3} \), \( \mathcal{L}\{e^{-3t}\} = \frac{3}{s+3} \), \( \mathcal{L}\{e^{-3t}\} = e^{-3t} \), etc.

Theorem: \( \mathcal{L}^{-1} \) is a linear transform.

\( \mathcal{L}^{-1}\{aF(s) + bg(s)\} = a\mathcal{L}^{-1}\{F(s)\} + b\mathcal{L}^{-1}\{g(s)\} \).

Proof: Suppose \( F(s) = \mathcal{L}\{f(t)\} \) and \( g(s) = \mathcal{L}\{g(t)\} \). Then \( \mathcal{L}\{F(s) + G(s)\} = \mathcal{L}\{f(t) + g(t)\} \).

\( \mathcal{L}^{-1}\{aF(s) + bg(s)\} = a\mathcal{L}^{-1}\{F(s)\} + b\mathcal{L}^{-1}\{g(s)\} = a\mathcal{L}^{-1}\{F(s)\} + b\mathcal{L}^{-1}\{G(s)\} \).
Review: Partial Fractions: Consider a rational function \( \frac{P(s)}{Q(s)} \) where \( P(s) \) and \( Q(s) \) are polynomials with real coefficients and \( \deg(Q(s)) > \deg(P(s)) \).

1. Factor and cancel common factors of \( P(s) \) and \( Q(s) \).

2. For each linear term \((s-a)^m\), \( a \in \mathbb{R} \) in the denominator, include terms of the form:
   \[
   \frac{A_1}{(s-a)} + \frac{A_2}{(s-a)^2} + \ldots + \frac{A_m}{(s-a)^m}.
   \]

3. For each irreducible quadratic term \([(s-a)^2 + b^2)^p\), \( a, b \in \mathbb{R} \), include terms of the form:
   \[
   \frac{B_1 s + C_1}{(s-a)^2 + b^2} + \frac{B_2 s + C_2}{(s-a)^2 + b^2} + \ldots + \frac{B_p s + C_p}{(s-a)^2 + b^2}.
   \]

4. Set \( P(s) \) equal to the sum of these terms.

5. Put over common denominator.


7. Find \( A_i, B_i, C_i \) by equating coefficients of \( s^k \) or \( s^0 \).

8. Evaluate both sides at the roots.
e.g. 7 Find \( \frac{s^2 + 5s + 7}{(s+2)(s^2+4)} \).

\[
\frac{A_1}{s+2} + \frac{B_1 s + C_1}{(s^2+4)} = \frac{A_1(s^2+4) + (B_1 s + C_1)(s+2)}{(s+2)(s^2+4)}
\]

Let
\[
s = \frac{A_1 s^2 + B_1 s^2 + 2B_1 s + C_1 s + 2C_1}{(s+2)(s^2+4)}.
\]

So
\[
\frac{s}{(s+2)(s^2+4)} = \frac{-1}{4(s+2)} + \frac{1}{2(s^2+4)}.
\]

Alternatively, at \( s = \pm 2 \) could have evaluated at roots:
\[
s = \pm 2: \quad s = 2A \quad \Rightarrow \quad A_1 = -\frac{1}{4}.
\]

Then
\[
-2B_1 + 4C_1 = 1 \quad \Rightarrow \quad B_1 = 1 - C_1
\]

\[
\Rightarrow \quad A_1 = -\frac{1}{2} + C_2
\]

Now consider \( \cos(\omega t) \) and \( \sin(\omega t) \).
Transform of a Derivative:

Theorem 7.2.2: If $F$ is $C^{n+1}$ on $[0, \infty)$ and $F, \ldots, F^{(n-1)}$ are of exponential order $t$ if $F^{(n)}(t)$ is piecewise continuous on $[0, \infty)$, then

$$\mathcal{L}[F^{(n)}(t)] = s^n F(s) - s^{n-1} F(0) - s^{n-2} F'(0) - \ldots - F^{(n-1)}(0),$$

where $F(0) = \mathcal{L}[F(t)]$.

$$e^{-st} F(t) = \int_0^\infty e^{-st} F(t) dt = e^{-st} F(t) \bigg|_0^\infty + \int_0^\infty e^{-st} F(t) e^{st} dt = F(0) + s \int_0^\infty e^{-st} F(t) dt = F(0) + sF(s).$$

Solving Linear IVP's w/ Constant Coef. using Laplace Transforms:

* Although 7.4 provides methods for doing this too, often IVP's are easier to solve using Laplace transforms.*

1. Apply $\mathcal{L}$ to both sides:

$$a_n s^n Y(s) + \ldots + a_0 Y(s) = g(t).$$

2. By Theorem 7.2.2, this becomes:

$$a_n [s^n Y(s) - s^{n-1} y(0) - \ldots - y^{(n-1)}(0)] + a_{n-1} [s^{n-1} Y(s) - s^{n-2} y(0) - \ldots - y^{(n-2)}(0)] + \ldots + a_0 Y(s) = G(s).$$

3. Solve for $Y(s)$. 

4. Apply $L^{-1}$. This will give you the solution $y(t)$ of the original IVP.

- $g(t)$, $g(t) = 0$, $y(0) = -3$.

1. $2y'' + 4y = 0$, $y(0) = 0$.
2. $2sY(s) - y_0 + Y(s) = 0$.
3. $2sY(s) + Y(s) = 0$.

E.g. $y'' + 4y = e^t$, $y(0) = 0$, $y'(0) = 0$.

- $2y'' + 4y = 8e^t$.
- $[sY(s) - y_0 - y'(0)] + 9y(s) = \frac{1}{s-1}$.
- $Y(s) = \frac{1}{(s^2+9)(s-1)} = \frac{A}{s-1} + \frac{Bs + C}{s^2 + 9}$.

At $s = 1$: $1 = 10A_1 = 7A_1 = \frac{1}{10}$. $1 = \frac{1}{10}s^2 + \frac{9}{10} + B_1s + C_1s - C_1$.

E.g. $1 = \left(\frac{1}{10} + B_1\right)s^2 + (C_1 - 8) + (90 - C_1) = 1 = \frac{9}{10} - C_1$.

$B_1 = -\frac{C_1}{10}$.

$Y(s) = \frac{1}{10(s-1)} + \frac{s}{10(s^2+9)} - \frac{1}{10(s^2+9)}$. 

- $Y(s)$ is the Laplace Transform of $y(t)$.
\[ y(t) = \frac{-1}{10} e^{-\frac{t}{2}} \left( \frac{5}{5t+3} \right) - \frac{1}{10} e^{-\frac{t}{2}} \left( \frac{5}{5t+3} \right) - \frac{1}{30} e^{-\frac{t}{2}} \left( \frac{9}{5t+3} \right) \]

\[ = \frac{-1}{10} e^{-\frac{t}{2}} \left( \frac{5}{5t+3} \right) - \frac{1}{30} \sin(3t) \]