Tutorial Info:

- **Tutorial Website:** http://ms.mcmaster.ca/~dedieula/2Z03.html

- **Office Hours:** Mondays 3pm - 5pm (in the Math Help Centre)
Tutorial #8:

- 10.2 Homogeneous Linear Systems
- 4.1 Definition of the Laplace Transform
- 4.2 The Inverse Transform and Transforms of Derivatives
10.2 Homogeneous Linear Systems

1. a) Find the general solution for the homogeneous system of linear DE’s:

\[ X' = \begin{pmatrix} 6 & -1 \\ 5 & 4 \end{pmatrix} X, \quad X(0) = \begin{bmatrix} -2 \\ 8 \end{bmatrix}. \]

Recall: Consider the homogeneous linear DE \( X' = AX \). If \( \lambda = \alpha + \beta i \) is an eigenvalue of the coefficient matrix \( A \), with corresponding eigenvector \( \nu = B_1 + B_2 i \), then two linearly independent solutions of this system on \((-\infty, \infty)\) are:

\[ X_1 = e^{\alpha t} \left[ B_1 \cos(\beta t) - B_2 \sin(\beta t) \right] \]
\[ X_2 = e^{\alpha t} \left[ B_1 \sin(\beta t) + B_2 \cos(\beta t) \right]. \]

b) Sketch the solution curve corresponding to this IVP.
10.2 Homogeneous Linear Systems

\[ x' = 6x - y \\
\[ y' = 5x + 4y \]
10.2 Homogeneous Linear Systems
10.2 Homogeneous Linear Systems
10.2 Homogeneous Linear Systems
2. Find the general solution for

\[ X' = \begin{pmatrix} 5 & -4 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 5 \end{pmatrix} X. \]
10.2 Homogeneous Linear Systems

- **Recall:** Consider the homogeneous linear system \( X' = AX \). If \( A \) has an eigenvalue \( \lambda \) of multiplicity \( m \) with only one corresponding eigenvector \( K \), then you can always find \( n \) linearly independent solutions of the form

\[
X_1 = Ke^{\lambda t}
\]

\[
X_2 = (Kt + P_1)e^{\lambda t}
\]

\[
X_3 = \left(\frac{t^2}{2}K + P_1t + P_2\right)e^{\lambda t}, \ldots \text{ etc.}
\]

\[
X_1 \text{ solution} \Rightarrow (A - \lambda I)K = 0
\]

\[
X_2 \text{ solution} \Rightarrow (A - \lambda I)K = 0 \text{ and } (A - \lambda I)P_1 = K
\]

\[
X_3 \text{ solution} \Rightarrow (A - \lambda I)K = 0 \text{ and } (A - \lambda I)P_1 = K \text{ and } (A - \lambda I)P_2 = P_1
\]
3. Find $\mathcal{L}\left\{(1 + e^{2t})^2\right\}$.

4. Find $\mathcal{L}\{t\}$ using the formal definition of the Laplace transform.

Recall: $\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t) dt$.

5. Find $\mathcal{L}^{-1}\left\{\frac{1}{s^2 + 3s}\right\}$.
Review: Partial Fractions

- Consider a rational function \( \frac{P(s)}{Q(s)} \), where \( P(s) \) and \( Q(s) \) are polynomials with real coefficients, and \( \text{deg}(P(s)) < \text{deg}(Q(s)) \).

1. Factor and cancel common factors of \( P(s) \) and \( Q(s) \).
2. For each linear term \((s - a)^m, a \in \mathbb{R}\), in the denominator, include terms of the form:
   \[
   \frac{A_1}{s-a} + \frac{A_2}{(s-a)^2} + \cdots + \frac{A_m}{(s-a)^m}.
   \]
3. For each irreducible quadratic term \([ (s - \alpha)^2 = \beta^2 ]^p, \alpha, \beta \in \mathbb{R}\), include terms of the form
   \[
   \frac{B_1s + C_1}{(s-\alpha)^2 + \beta^2} + \frac{B_2s + C_2}{((s-\alpha)^2 + \beta^2)^2} + \cdots + \frac{B_p s + C_p}{((s-\alpha^2)^p + \beta^2)^p}.
   \]
Review: Partial Fractions

4. Set $\frac{P(s)}{Q(s)}$ equal to the sum of these terms.

5. Put over common denominator.


7. a) Find $A_i, B_i, C_i$ by equating coefficients $s^k$.

   b) Evaluate both sides at the roots.
4.1/4.2 The Laplace Transform

- 6. Use the Laplace transform to solve the linear IVP

\[ 2y' + y = 0, \quad y(0) = -3. \]

- **Recall:** To solve this we want to \( \mathcal{L} \) both sides, isolate for \( \mathcal{L} \{y\} := Y(s) \), then \( \mathcal{L}^{-1} \) both sides.
Table of Laplace Transforms:

Here $\mathcal{L}\{f(t)\} = F(s)$.

<table>
<thead>
<tr>
<th>Transforms of Some Basic Functions</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathcal{L}{1} = \frac{1}{s}$</td>
</tr>
<tr>
<td>$\mathcal{L}{t^n} = \frac{n!}{s^{n+1}}, n = 1, 2, \ldots$</td>
</tr>
<tr>
<td>$\mathcal{L}{e^{at}} = \frac{1}{s-a}$</td>
</tr>
<tr>
<td>$\mathcal{L}{\sin(kt)} = \frac{k}{s^2 + k^2}$</td>
</tr>
<tr>
<td>$\mathcal{L}{\cos(kt)} = \frac{s}{s^2 + k^2}$</td>
</tr>
<tr>
<td>$\mathcal{L}{\sinh(kt)} = \frac{k}{s^2 - k^2}$</td>
</tr>
<tr>
<td>$\mathcal{L}{\cosh(kt)} = \frac{s}{s^2 - k^2}$</td>
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<thead>
<tr>
<th>Transforms of Derivatives</th>
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<tbody>
<tr>
<td>$\mathcal{L}{f^{(n)}(t)} = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \ldots - f^{(n-1)}(0)$</td>
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</tbody>
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<thead>
<tr>
<th>Translation Theorems</th>
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<tbody>
<tr>
<td>$\mathcal{L}{e^{at}f(t)} = \mathcal{L}{f(t)} \mid_{s \to s-a} = F(s-a), \text{ where } a \in \mathbb{R}$</td>
</tr>
<tr>
<td>$\mathcal{L}^{-1}{F(s-a)} = \mathcal{L}^{-1}{F(s)} \mid_{s \to s-a} = e^{at}f(t)$</td>
</tr>
<tr>
<td>$\mathcal{L}{f(t-a)U(t-a)} = e^{-as}F(s)$, where $a &gt; 0$</td>
</tr>
<tr>
<td>$\mathcal{L}^{-1}{e^{-as}F(s)} = f(t-a)U(t-a)$, where $a &gt; 0$</td>
</tr>
<tr>
<td>$\mathcal{L}{g(t)U(t-a)} = e^{-as} \mathcal{L}{g(t+a)}$, where $a &gt; 0$</td>
</tr>
<tr>
<td>$\mathcal{L}{U(t-a)} = \frac{e^{-as}}{s}$, where $a &gt; 0$</td>
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<thead>
<tr>
<th>Derivatives of Transforms &amp; Convolution</th>
</tr>
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<tbody>
<tr>
<td>$\mathcal{L}{t^n f(t)} = (-1)^n \frac{d^n}{ds^n} F(s), n = 1, 2, \ldots$</td>
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<tr>
<td>$\mathcal{L}{f \ast g} = \mathcal{L}{f(t)} \mathcal{L}{g(t)} = F(s)G(s)$</td>
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<th>Dirac Delta Function</th>
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<tr>
<td>$\mathcal{L}{\delta(t-t_0)} = e^{-as}, \text{ for } t_0 &gt; 0$</td>
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