Tutorial Info:

- **Tutorial Website:** http://ms.mcmaster.ca/~dedieula/2Z03.html

- **Office Hours:** Mondays 3pm - 5pm (in the Math Help Centre)
Tutorial #6:

- 3.4 Undetermined Coefficients
- 3.5 Variation of Parameters
3.4 Undetermined Coefficients

- **Undetermined Coefficients**: a method of solution for finding a particular solution to a linear differential equation with *constant coefficients*

\[ a_n y^{(n)} + \cdots + a_1 y' + a_0 y = g(x), \]

where \( g(x) \) is a polynomial, exponential \( e^{ax} \), sine, cosine, or some sum/product of these function.
### 3.4 Undetermined Coefficients

**Method of Solution:**

1. Find the general solution, $y_c$, to the associated homogeneous equation

$$a_n y^{(n)} + \cdots + a_1 y' + a_0 y = g(x).$$

2. Choose a trial particular solution, $y_{p_i}$ for each term in $g(x)$.

3. If any $y_{p_i}$ contains terms that duplicate terms in $y_c$, then multiply $y_{p_i}$ by $x^n$, where $n$ is the smallest possible integer that eliminate the duplication.

4. Plug the sum of the terms found in Step 3, $y_p$, into the original DE and solve for the undetermined coefficients.

5. The general solution is $y = y_c + y_p$. 
3.4 Undetermined Coefficients

- **Trial Particular Solutions:**
  - A polynomial of degree $k$ corresponds to a trial solution $a_0 + a_1 x + \cdots + a_k x^k$.
  - $e^{mx}$ corresponds to trial solution $Ae^{mx}$.
  - $\sin(mx)$ and $\cos(mx)$ both correspond to the trial solution $A\cos(mx) + B\sin(mx)$.
  - If you have a product of the above, then take the corresponding product of trial solutions.
    - **e.g.** $xe^{3x}\cos(4x)$ corresponds to trial solution
      $$(a_0 + a_1 x)(Ae^{3x})(B\cos(4x) + C\sin(4x)) = (c_0 + c_1 x)(e^{3x})(B\cos(4x) + C\sin(4x)).$$
3.4 Undetermined Coefficients

1. For each of the following, the general solution of the associated homogenous equation is given. What form will its particular solution have?
   a) $y'' + 3y = -48x^2e^{3x}; y_c = c_1\cos(\sqrt{3}x) + c_2\sin(\sqrt{3}x)$
   b) $y''' - 6y'' = 3 - \cos x; y_c = c_1 + c_2x + c_3e^{6x}$
   c) $y'' - y' + \frac{1}{4}y = 3 + e^{\frac{x}{2}}; y_c = c_1e^{\frac{x}{2}} + c_2xe^{\frac{x}{2}}$

2. Find the general solution of the differential equations in #1.
3.5 Variation of Parameters

- **Variation of Parameters:** a method of solution for finding a particular solution to a linear differential equation

\[ a_n(x)y^{(n)} + \cdots + a_1(x)y' + a_0(x)y = g(x). \]

To use it, we must already have a general solution for the corresponding homogeneous equation

\[ a_n(x)y^{(n)} + \cdots + a_1(x)y' + a_0(x)y = 0 \]

- **Advantages Over Method of Undetermined Coefficients:**
  - For Undetermined Coefficients, we require that \( g(x) \) is a sum/product of polynomials, \( e^{ax} \)'s, sines, and cosines. For Variation of Parameters, we require no such restrictions on \( g(x) \).
  - Undetermined Coefficients requires that the linear DE has constant coefficients, whereas Variation of Parameters does not.
3.5 Variation of Parameters

- **Method of Solution**: I write the method of solution for second-order linear DE’s, but this method naturally generalizes for higher order DE’s.

1. Find the general solution, \( y_c = c_1 y_1 + c_2 y_2 \), for corresponding homogeneous equation \( a_2(x)y'' + a_1(x)y' + a_0(x)y = 0 \).
2. Compute Wronskian \( W(y_1(x), y_2(x)) \).
3. Put equation in standard form \( y'' + P(x)y' + Q(x)y = f(x) \).
4. Compute \( W_1 = \begin{vmatrix} 0 & y_2 \\ f(x) & y'_2 \end{vmatrix}, \quad W_2 = \begin{vmatrix} y_1 & 0 \\ y'_1 & f(x) \end{vmatrix} \).
5. Find \( u_1 := \int \frac{W_1}{W} \, dx \) and \( u_2 := \int \frac{W_2}{W} \, dx \).
6. A particular solution is \( y_p = u_1 y_1 + u_2 y_2 \), and the general solution is \( y = y_c + y_p \).
3.5 Variation of Parameters

3. Find the general solution of

\[ x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right)y = x^\frac{3}{2}, \]

where \( y_1 = x^{-\frac{1}{2}}\cos x \) and \( y_2 = x^{-\frac{1}{2}}\sin x \) and linearly independent solutions of the associated homogeneous DE on \((0, \infty)\).

4. Solve \( y''' + y' = \tan x \).