1 Exercise 1: Geometric series and a bit more

1.1 Problem

Let \( a \in \mathbb{Q} \) and \( b \in \mathbb{Q} \). Prove that the equalities

\[
(a - b) \left( a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \cdots + ab^{n-2} + b^{n-1} \right) = a^n - b^n \tag{1}
\]

and

\[
(a - b)^2 \left( 1a^{n-1} + 2a^{n-2}b + 3a^{n-3}b^2 + \cdots + (n - 1) ab^{n-2} + nb^{n-1} \right) = a^{n+1} - (n + 1) ab^n + nb^{n+1} \tag{2}
\]

hold for each \( n \in \mathbb{N} \).

(Here and in the following, \( \mathbb{N} \) stands for the set \( \{0, 1, 2, \ldots\} \). We also recall that empty sums – i.e., sums that have no addends at all – evaluate to 0 by definition. This applies, in particular, to the sums \( a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \cdots + ab^{n-2} + b^{n-1} \) and \( 1a^{n-1} + 2a^{n-2}b + 3a^{n-3}b^2 + \cdots + (n - 1) ab^{n-2} + nb^{n-1} \) in the case when \( n = 0 \).)
1.2 Remark

A consequence of the formulas [1] and [2] is that every rational number \( x \neq 1 \) satisfies

\[
1 + x + x^2 + \cdots + x^{n-1} = \frac{1 - x^n}{1 - x} \quad \text{and} \quad 1 + 2x + 3x^2 + \cdots + nx^{n-1} = \frac{1 - (n + 1)x^n + nx^{n+1}}{(1 - x)^2}.
\]

Indeed, these equalities follow by setting \( a = 1 \) and \( b = x \) in the equalities [1] and [2] and dividing by \( 1 - x \) or \( 1 - x^2 \), respectively.

More generally, the formulas [1] and [2] remain true when \( a \) and \( b \) are two commuting elements of an arbitrary ring (we will later learn what this means; for now, let us just say that, e.g., we could let \( a \) and \( b \) be two commuting matrices instead of rational numbers).

1.3 Solution

We will use the summation sign when we solve this exercise. This will make our formulas both shorter and clearer. For example, instead of “\( 1a^{n-1} + 2a^{n-2}b + 3a^{n-3}b^2 + \cdots + (n - 1)ab^{n-2} + nb^{n-1} \)”, it will let us just write “\( \sum_{k=1}^{n} kb^{k-1} \)”.

Let us give a crash course on the use of the summation sign. We refer to [Grinbe19, Section 1.4] for details and further information.

- Assume that \( S \) is a finite set, and that \( a_s \) is a number (e.g., a real number) for each \( s \in S \). Thus you have \( |S| \) many numbers \( a_s \) in total. Then, \( \sum_{s \in S} a_s \) shall denote the sum of all of these \( |S| \) many numbers. For example,

\[
\sum_{s \in \{2,5,6\}} s^3 = 2^3 + 5^3 + 6^3
\]

(here, \( S = \{2,5,6\} \) and \( a_s = s^3 \) for each \( s \in S \)) and

\[
\sum_{s \in \{5,7,9,11\}} \frac{1}{s} = \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11}
\]

(here, \( S = \{5,7,9,11\} \) and \( a_s = \frac{1}{s} \) for each \( s \in S \)).

The letter \( s \) here plays the same role as the letter \( s \) in “\( \{s^2 \mid s \in \{2,3,4\}\} \)” or in “the function that sends each integer \( s \) to \( s^2 - 1 \)”; it designates the “moving part” in a definition (it is what is called a “bound variable” or a “running index”). You don’t have to use the specific letter \( s \) for it; you can use any other letter instead (as long as it does not already have a different meaning) and get the same result. For example, the sum \( \sum_{s \in \{2,5,6\}} s^3 \) can be rewritten as \( \sum_{i \in \{2,5,6\}} i^3 \) or as \( \sum_{\varnothing \in \{2,5,6\}} \varnothing^3 \). When the set \( S \) is empty (so you have no numbers \( a_s \) at all), the sum \( \sum_{s \in S} a_s \) is defined to be 0; this is called an empty sum.

\[1\text{ and to [Grinbe19] Section 2.14} \] for proofs of well-definedness and basic properties.
Assume that \(u\) and \(v\) are two integers, and that \(a_s\) is a number (e.g., a real number) for each \(s \in \{u, u+1, \ldots, v\}\). (When \(u > v\), we understand the set \(\{u, u+1, \ldots, v\}\) to be empty – it does not contain any “anti-integers” either.) Then, \(\sum_{s=u}^{v} a_s\) is just a shorthand for the sum \(\sum_{s \in \{u, u+1, \ldots, v\}} a_s\). This sum can also be written as \(a_u + a_{u+1} + \cdots + a_v\), but this notation presumes the reader to guess what the “general term” \(a_s\) looks like. For example,

\[
\sum_{s=5}^{10} s^s = 5^5 + 6^6 + 7^7 + 8^8 + 9^9 + 10^{10} = 5^5 + 6^6 + \cdots + 10^{10}
\]

(arguably, guessing the general term is easy here, but look at the sum in (2)). For another example,

\[
\sum_{s=-2}^{2} s^2 = (-2)^2 + (-1)^2 + 0^2 + 1^2 + 2^2.
\]

Expressions of the form \(\sum_{s \in S} a_s\) and \(\sum_{s \in \{u, u+1, \ldots, v\}} a_s\) are called “finite sums”, and the \(\sum\) symbol is called the “summation sign”.

Finite sums satisfy the rules that you would expect. For example, assume that a finite set \(S\) is written as a union of two disjoint subsets \(A\) and \(B\) (so each element of \(S\) belongs to one of \(A\) and \(B\), but not to both). Assume that \(a_s\) is a number for each \(s \in S\). Then,

\[
\sum_{s \in S} a_s = \sum_{s \in A} a_s + \sum_{s \in B} a_s.
\]

For example, if \(S = \{1, 2, \ldots, 2n\}\) for some \(n \in \mathbb{N}\), and if

\[
A = \{\text{the even elements of } S\} = \{2, 4, 6, \ldots, 2n\}
\]
and

\[
B = \{\text{the odd elements of } S\} = \{1, 3, 5, \ldots, 2n-1\},
\]

then this formula becomes

\[
a_1 + a_2 + \cdots + a_{2n} = (a_2 + a_4 + a_6 + \cdots + a_{2n}) + (a_1 + a_3 + a_5 + \cdots + a_{2n-1}).
\]

This is exactly what you would expect: To sum the \(2n\) numbers \(a_1, a_2, \ldots, a_{2n}\), you can first split them into the “even” and the “odd” ones (to be pedantic: rather, the ones with the even subscripts and the ones with the odd subscripts), and separately sum the former and the latter, and subsequently add the two small sums together. See [Grinbe19 Section 1.4.2] for this and several other rules (and for their rigorous proofs, if you are that skeptical). You can use all these rules without saying, except for the “telescoping sums” rule (which you should cite by name when you apply it). For lots of practice with sums, see [GrKnPa94, Chapter 2 and further].

The “product sign” \(\prod\) is analogous to the summation sign \(\sum\), but stands for products instead of sums. For example,

\[
\prod_{s=5}^{10} s^s = 5^5 \cdot 6^6 \cdot 7^7 \cdot 8^8 \cdot 9^9 \cdot 10^{10} = 5^5 \cdot 6^6 \cdot \cdots \cdot 10^{10}.
\]

An empty product (i.e., a product of the form \(\prod_{s \in S} a_s\) when \(S\) is empty) is defined to be 1. See [Grinbe19 Section 1.4.4] for the properties of products.
The summation sign lets us rewrite the sum $a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \cdots + ab^{n-2} + b^{n-1}$ in (1) as $\sum_{k=1}^{n} a^{n-k}b^{k-1}$, and lets us rewrite the sum $1a^{n-1} + 2a^{n-2}b + 3a^{n-3}b^2 + \cdots + (n-1) ab^{n-2} + nb^{n-1}$ in (2) as $\sum_{k=1}^{n} ka^{n-k}b^{k-1}$. So the two equalities (1) and (2) rewrite as

\[(a - b) \sum_{k=1}^{n} a^{n-k}b^{k-1} = a^n - b^n\]  \hspace{1cm} (3)

and

\[(a - b)^2 \sum_{k=1}^{n} ka^{n-k}b^{k-1} = a^{n+1} - (n + 1) ab^n + nb^{n+1},\]  \hspace{1cm} (4)

respectively. It is in these forms that we will prove these equalities.

- **Proof of (3):**
  We shall prove (3) by induction on $n$:

  **Induction base:** Comparing the equalities $a^0 - b^0 = 1^0 - 1^0 = 0$ and

  \[(a - b) \sum_{k=1}^{0} a^{0-k}b^{k-1} = (a - b) 0 = 0,\]

  we obtain

  \[(a - b) \sum_{k=1}^{0} a^{0-k}b^{k-1} = a^0 - b^0.\]

  In other words, (3) holds for $n = 0$. Thus the induction base is complete.

  **Induction step:** Let $m \in \mathbb{N}$. Assume that (3) holds for $n = m$. We must prove that (3) holds for $n = m + 1$.

  We have assumed that (3) holds for $n = m$. In other words, we have

  \[(a - b) \sum_{k=1}^{m} a^{m-k}b^{k-1} = a^m - b^m.\]  \hspace{1cm} (5)

  Now, splitting off the last addend of the sum $\sum_{k=1}^{m+1} a^{(m+1)-k}b^{k-1}$, we obtain

  \[
  \sum_{k=1}^{m+1} a^{(m+1)-k}b^{k-1} = \sum_{k=1}^{m} a^{(m+1)-k}b^{k-1} + a^{(m+1)-(m+1)}b^{(m+1)-(m+1)} = a^0 = 1 = a^{m+1} = \sum_{k=1}^{m} a^{m-k}b^{k-1} + b^m = a \sum_{k=1}^{m} a^{m-k}b^{k-1} + b^m,\]

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so that

\[(a - b) \sum_{k=1}^{m+1} a^{m+1-k}b^{k-1}\]

\[= (a - b) \left( a \sum_{k=1}^{m} a^{m-k}b^{k-1} + b^{m} \right)\]

\[= a (a - b) \sum_{k=1}^{m} a^{m-k}b^{k-1} + (a - b) b^{m} = a (a^{m} - b^{m}) + (a - b) b^{m}\]

\[= aa^{m} - ab^{m} + ab^{m} - bb^{m} = a_{m+1}^{m+1} - b_{m+1}^{m+1} = a^{m+1} - b^{m+1}.\]

In other words, \((3)\) holds for \(n = m + 1\). This completes the induction step. Thus, \((3)\) is proven.

**Proof of \((4)\):**

We shall prove \((4)\) by induction on \(n\):

**Induction base:** Comparing the equalities \(a^{0+1} - (0 + 1) ab^{0} + 0b^{0+1} = a^{1} - a = a - a = 0\) and

\[(a - b)^{2} \sum_{k=1}^{0} ka^{0-k}b^{k-1} = (a - b)^{2} 0 = 0,\]

we obtain

\[(a - b)^{2} \sum_{k=1}^{0} ka^{0-k}b^{k-1} = a^{0+1} - (0 + 1) ab^{0} + 0b^{0+1}.\]

In other words, \((4)\) holds for \(n = 0\). Thus the induction base is complete.

**Induction step:** Let \(m \in \mathbb{N}\). Assume that \((4)\) holds for \(n = m\). We must prove that \((4)\) holds for \(n = m + 1\).

We have assumed that \((4)\) holds for \(n = m\). In other words, we have

\[(a - b)^{2} \sum_{k=1}^{m} ka^{m-k}b^{k-1} = a^{m+1} - (m + 1) ab^{m} + mb^{m+1}.\] (6)

Now, splitting off the last addend of the sum \(\sum_{k=1}^{m+1} ka^{(m+1)-k}b^{k-1}\), we obtain

\[\sum_{k=1}^{m+1} ka^{(m+1)-k}b^{k-1} = \sum_{k=1}^{m} ka^{(m+1)-k}b^{k-1} + (m + 1) a_{m+1}^{m+1} - b_{m+1}^{m+1}\]

\[= \sum_{k=1}^{m} ka a^{m-k}b^{k-1} + (m + 1) b^{m} = a \sum_{k=1}^{m} ka^{m-k}b^{k-1} + (m + 1) b^{m},\]
so that

\[(a - b)^2 \sum_{k=1}^{m+1} ka^{m+1-k}b^{k-1}\]

\[= (a - b)^2 \left( a \sum_{k=1}^{m} ka^{m-k}b^{k-1} + (m + 1) b^m \right)\]

\[= a (a - b)^2 \sum_{k=1}^{m} ka^{m-k}b^{k-1} + (a - b)^2 (m + 1) b^m\]

\[= a^{m+2} - (m + 1) a^2 b^m + mab^{m+1} + (a - b)^2 (m + 1) b^m\]

\[= a^{m+2} - (m + 1) a^2 b^m + mab^{m+1} + (m + 1) a^2 b^m - 2 (m + 1) a b^{m+1} + (m + 1) b^2 b^m\]

\[= a^{m+2} + mab^{m+1} - 2 (m + 1) a b^{m+1} + (m + 1) b^2 b^m\]

\[= a^{m+2} + mab^{m+1} - 2 (m + 1) a b^{m+1} + (m + 1) b^2 b^m\]

\[= a^{m+2} - (m + 2) a b^{m+1} + (m + 1) b^m + 2\]

\[= a^{m+1} - ((m + 1) + 1) a b^{m+1} + (m + 1) b^m + 2\]

In other words, (4) holds for \( n = m + 1 \). This completes the induction step. Thus, (4) is proven.

So the exercise is solved.

1.4 Remark

The equality (3) can also be proved using the telescope principle; see [Grinbe19, (18)] for this argument.

2 Exercise 2: Factorials 101

2.1 Problem

Recall that the factorial of a nonnegative integer \( n \) is defined by

\[ n! = \prod_{i=1}^{n} i = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot n. \]
Thus, in particular, \(0! = 1\) (since we defined empty products to be 1); it is easy to see that
\[
1! = 1, \quad 2! = 2, \quad 3! = 6, \quad 4! = 24, \quad 5! = 120, \quad 6! = 720, \quad 7! = 5040.
\]

This sequence grows very fast (see Stirling’s approximation).

Prove the following properties of factorials:

(a) We have \(n! = n \cdot (n - 1)!\) for each positive integer \(n\).

(b) For each \(n \in \mathbb{N}\), we have
\[
1 \cdot 1! + 2 \cdot 2! + \cdots + n \cdot n! = (n + 1)! - 1.
\]

(c) For each \(n \in \mathbb{N}\), we have
\[
1 \cdot 3 \cdot 5 \cdots (2n - 1) = \frac{(2n)!}{2^n n!}.
\]

(Here, the left hand side is understood to be the product of the first \(n\) odd positive integers, i.e., the product \(\prod_{i=1}^{n} (2i - 1)\).)

2.2 Solution

(a) Let \(n\) be a positive integer. Thus, \(n \in \{1, 2, \ldots, n\}\). The definition of \((n - 1)!\) yields
\[
(n - 1)! = \prod_{i=1}^{n-1} i.
\]

But the definition of \(n!\) yields
\[
n! = \prod_{i=1}^{n} i = \left( \prod_{i=1}^{n-1} i \right) \cdot n
\]

(here, we have split off the factor for \(i = n\) from the product, since \(n \in \{1, 2, \ldots, n\}\)). Hence,
\[
n! = \left( \prod_{i=1}^{n-1} i \right) \cdot n = (n - 1)! \cdot n = n (n - 1)!
\]

(by (7))

This solves part (a) of the exercise.

(b) Claims like this can often be proven in two ways: by (fairly straightforward) induction, and by (usually tricky) transformations. In this particular case, the two proofs are actually very similar, and can easily be transformed into one another; nevertheless, let us show both of them.

Proof by induction: We shall prove the claim of part (b) by induction on \(n\):

Induction base: We have
\[
1 \cdot 1! + 2 \cdot 2! + \cdots + 0 \cdot 0! = (\text{empty sum}) = 0.
\]
Comparing this with \((0+1)!-1 = 1-1 = 0\), we obtain \(1\cdot 1! + 2\cdot 2! + \cdots + m \cdot m! = (0+1)!-1\).

Thus, the claim of part (b) holds for \(n = 0\). This completes the induction base.

**Induction step:** Let \(m \in \mathbb{N}\). Assume that the claim of part (b) holds for \(n = m\). We must prove that the claim of part (b) holds for \(n = m+1\).

We have assumed that the claim of part (b) holds for \(n = m\). In other words, we have

\[
1 \cdot 1! + 2 \cdot 2! + \cdots + m \cdot m! = (m+1)! - 1.
\]

Now,

\[
1 \cdot 1! + 2 \cdot 2! + \cdots + (m+1) \cdot (m+1)! = \underbrace{(1 \cdot 1! + 2 \cdot 2! + \cdots + m \cdot m!)}_{=(m+1)!-1} + (m+1) \cdot (m+1)!
\]

\[
= (m+1)! - 1 + (m+1) \cdot (m+1)!
\]

\[
= (1 + (m+1)) \cdot (m+1)! - 1
\]

\[
= (m+2) \cdot (m+1)! - 1.
\]

But part (a) of this exercise (applied to \(n = m+2\)) yields

\[
(m+2)! = (m+2) \cdot (m+2) - (m+1)! = (m+2) \cdot (m+1)!.
\]

Hence, (8) becomes

\[
1 \cdot 1! + 2 \cdot 2! + \cdots + (m+1) \cdot (m+1)! = \underbrace{(m+2)!}_{=\underbrace{(m+1)!}_{=(m+1+1)!}} - 1 = ((m+1)+1)! - 1.
\]

In other words, the claim of part (b) holds for \(n = m + 1\). This completes the induction step. Thus, the claim of part (b) is proven by induction.

**Proof by tricky transformations:** This proof shall rely on the following fact:

**Proposition 2.1.** Let \(m \in \mathbb{N}\). Let \(a_0, a_1, \ldots, a_m\) be \(m+1\) real numbers\(^2\). Then,

\[
\sum_{i=1}^{m} (a_i - a_{i-1}) = a_m - a_0.
\]

Proposition 2.1 is known as the “telescope principle” since it contracts the sum \(\sum_{i=1}^{m} (a_i - a_{i-1})\) to the single difference \(a_m - a_0\), like folding a telescope.

The simplest way to convince yourself that Proposition 2.1 is true is by expanding the left hand side:

\[
\sum_{i=1}^{m} (a_i - a_{i-1}) = (a_1 - a_0) + (a_2 - a_1) + (a_3 - a_2) + \cdots + (a_m - a_{m-1})
\]

\(^2\)I am saying “real numbers” just for the sake of saying something definite. You could just as well state this principle for “complex numbers” or “rational numbers” or (once we have learnt what an abelian group is) “elements of an abelian group (where the operation of the group is written as addition)”; the proof will be the same in each case.
and watching all the terms cancel each other out except for the \(-a_0\) and the \(a_m\). More formally, this argument can be emulated by an induction on \(m\). See [18f-hw0s, proof of Proposition 2.2] or [Grinbe19, proof of (16)] for formal proofs of Proposition 2.1.

Now, how can we apply Proposition 2.1 to part (b) of the exercise? We have \(1 \cdot 1! + 2 \cdot 2! + \cdots + n \cdot n! = \sum_{i=1}^{n} i \cdot i!\). If we could write each addend \(i \cdot i!\) in the form \(a_i - a_i - 1\) for some \(n + 1\) real numbers \(a_0, a_1, \ldots, a_n\), then we could use Proposition 2.1.

The tricky part is finding these \(a_i\). Namely, set \(a_i = (i+1)!\) for each \(i \in \{0, 1, \ldots, n\}\). Then, I claim that

\[
i \cdot i! = a_i - a_i - 1 \quad \text{for each } i \in \{1, 2, \ldots, n\}. \tag{9}
\]

The proof of (9) is not tricky at all: Let \(i \in \{1, 2, \ldots, n\}\). Then, part (a) of the exercise (applied to \(i+1\) instead of \(n\)) yields

\[
(i+1)! = (i+1) \cdot \left(\frac{(i+1) - 1}{i}\right)! = (i+1) \cdot i! = i \cdot i! + i!.
\]

Solving this for \(i \cdot i!\), we find

\[
i \cdot i! = (i+1)! - i!.
\]

Comparing this with

\[
a_i = \frac{a_i}{(i+1)!} = \frac{a_i - a_{i-1}}{(i+1)!} = (i+1)! - \left(\frac{(i-1) + 1}{i}\right)! = (i+1)! - i!,
\]

we obtain \(i \cdot i! = a_i - a_i - 1\). This proves (9).

Now,

\[
1 \cdot 1! + 2 \cdot 2! + \cdots + n \cdot n! = \sum_{i=1}^{n} i \cdot i! = \sum_{i=1}^{n} (a_i - a_i - 1) = a_n - a_0 = (n+1)! - 0! = (n+1)! - 1.
\]

This solves part (b) of the exercise again.

(c) Again, we give two proofs:

Proof by induction: We shall prove the claim of part (c) by induction on \(n\):

Induction base: We have

\[
1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2 \cdot 0 - 1) = (\text{empty product}) = 1.
\]

Comparing this with \(\frac{(2 \cdot 0)!}{2^{0}!} = \frac{0!}{1 \cdot 0!} = 1\), we obtain \(1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2 \cdot 0 - 1) = \frac{(2 \cdot 0)!}{2^{0}!}\). Thus, the claim of part (c) holds for \(n = 0\). This completes the induction base.
Induction step: Let $m \in \mathbb{N}$. Assume that the claim of part (c) holds for $n = m$. We must prove that the claim of part (c) holds for $n = m + 1$.

We have assumed that the claim of part (c) holds for $n = m$. In other words, we have

$$1 \cdot 3 \cdot 5 \cdots (2m - 1) = \frac{(2m)!}{2^m m!}.$$  \hspace{1cm} (10)

Our goal is to show that

$$1 \cdot 3 \cdot 5 \cdots (2 (m + 1) - 1) = \frac{(2 (m + 1))!}{2^{m+1} (m + 1)!}.$$  \hspace{1cm} (11)

We start by rewriting the factorials on the right hand side of this alleged equality in terms of the factorials in (10). Clearly, $2 (m + 1)$ is a positive integer. Hence, part (a) of the exercise (applied to $n = 2 (m + 1)$) yields

$$(2 (m + 1))! = 2 (m + 1) \cdot \left( \frac{2 (m + 1) - 1}{2m + 1} \right)! = 2 (m + 1) \cdot \left( \frac{2m + 1}{2m} \right)! = 2 (m + 1) \cdot (2m + 1) \cdot (2m)!.$$  \hspace{1cm} (12)

Also, part (a) of the exercise (applied to $n = m + 1$) yields

$$(m + 1)! = (m + 1) \cdot \left( \frac{m + 1 - 1}{m} \right)! = (m + 1) \cdot m!.$$  \hspace{1cm} (13)

Plugging the two equalities (12) and (13) as well as the obvious equality $2^{m+1} = 2 \cdot 2^m$ into the expression $\frac{(2 (m + 1))!}{2^{m+1} (m + 1)!}$, we obtain

$$\frac{(2 (m + 1))!}{2^{m+1} (m + 1)!} = \frac{2 (m + 1) \cdot (2m + 1) \cdot (2m)!}{(2 \cdot 2^m) (m + 1) \cdot m!} = \frac{(2m)!}{2^m m!} \cdot (2m + 1).$$

Comparing this with

$$1 \cdot 3 \cdot 5 \cdots (2 (m + 1) - 1) = \left( \frac{1 \cdot 3 \cdot 5 \cdots (2m - 1)}{2^m m!} \right) \cdot \left( \frac{2 (m + 1) - 1}{2m + 1} \right)$$

$$= \frac{(2m)!}{2^m m!} \cdot \frac{(2m)!}{(2m + 1)!}$$

$$= \frac{(2m)!}{2^m m!} \cdot (2m + 1),$$

we obtain precisely the equality (11) that we were trying to prove. In other words, the claim of part (c) holds for $n = m + 1$. This completes the induction step. Thus, the claim of part (c) is proven by induction.
Proof by tricky transformations: Let \( n \in \mathbb{N} \). This time, the trick is to split the product \((2n)! = 1 \cdot 2 \cdots (2n)\) into two smaller products – one containing all its even factors and one containing its odd factors. This yields

\[
(2n)! = 1 \cdot 2 \cdots (2n) = (2 \cdot 4 \cdot 6 \cdots (2n)) \cdot (1 \cdot 3 \cdot 5 \cdots (2n - 1))
\]

(here, we have factored out a 2 from each factor)

\[
= 2^n \cdot (1 \cdot 2 \cdot 3 \cdots n) \cdot (1 \cdot 3 \cdot 5 \cdots (2n - 1))
\]

(since \( n! = 1 \cdot 2 \cdots n \))

\[
= 2^n n! \cdot (1 \cdot 3 \cdot 5 \cdots (2n - 1))
\]

Solving this equation for \( 1 \cdot 3 \cdot 5 \cdots (2n - 1) \), we obtain

\[
1 \cdot 3 \cdot 5 \cdots (2n - 1) = \frac{(2n)!}{2^n n!}.
\]

Thus, part (c) is solved again.

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### 3 EXERCISE 3: BINOMIAL COEFFICIENTS 101

3.1 PROBLEM

For any \( n \in \mathbb{Q} \) and \( k \in \mathbb{N} \), we define the binomial coefficient \( \binom{n}{k} \) by

\[
\binom{n}{k} = \frac{n(n-1)(n-2) \cdots (n-k+1)}{k!} = \frac{\prod_{i=0}^{k-1} (n-i)}{k!}.
\]

We furthermore set \( \binom{n}{k} = 0 \) for all rational \( k \notin \mathbb{N} \).

For example,

\[
\binom{5}{3} = \frac{5 \cdot 4 \cdot 3}{3!} = \frac{60}{6} = 10;
\]

\[
\binom{1}{3} = \frac{1 \cdot 0 \cdot (-1)}{3!} = \frac{0}{6} = 0;
\]

\[
\binom{-2}{3} = \frac{(-2) \cdot (-3) \cdot (-4)}{3!} = \frac{-24}{6} = -4;
\]

\[
\binom{1/2}{3} = \frac{(1/2) \cdot (-1/2) \cdot (-3/2)}{3!} = \frac{3/8}{6} = \frac{1}{16};
\]

\[
\binom{4}{1/2} = 0 \quad \text{(since } 1/2 \notin \mathbb{N}).
\]

Prove the following properties of binomial coefficients:
(a) If \( n \in \mathbb{N} \) and \( k \in \mathbb{N} \) are such that \( n \geq k \), then
\[
\binom{n}{k} = \frac{n!}{k!(n-k)!}.
\]
(This is often used as a definition of the binomial coefficients, but it is a lousy definition, as it only covers the case when \( n, k \in \mathbb{N} \) and \( n \geq k \).)

(b) If \( n \in \mathbb{N} \) and \( k \in \mathbb{Q} \) are such that \( k > n \), then
\[
\binom{n}{k} = 0.
\]

(c) If \( n \in \mathbb{N} \) and \( k \in \mathbb{Q} \), then
\[
\binom{n}{k} = \binom{n}{n-k}.
\]
(This is known as the \textit{symmetry of binomial coefficients}. Note that it fails if \( n \notin \mathbb{N} \).)

(d) Any \( n \in \mathbb{Q} \) and \( k \in \mathbb{Q} \) satisfy
\[
\binom{-n}{k} = (-1)^k \binom{k+n-1}{k}.
\]
(This is one of the versions of the \textit{upper negation formula}.)

(e) Any \( n \in \mathbb{Q} \) and \( k \in \mathbb{Q} \) satisfy
\[
\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}.
\]
(This is the \textit{recurrence of the binomial coefficients}, and is the reason why each entry of \textit{Pascal's triangle} is the sum of the two entries above it.)

(f) Any \( n \in \mathbb{Q} \) and \( k \in \mathbb{Q} \) satisfy
\[
k\binom{n}{k} = n\binom{n-1}{k-1}.
\]

3.2 Solution

(a) Let \( n \in \mathbb{N} \) and \( k \in \mathbb{N} \) be such that \( n \geq k \). From \( k \in \mathbb{N} \), we obtain \( k \geq 0 \), thus \( n-k \leq n \). Combining this with \( n-k \geq 0 \) (since \( n \geq k \)), we obtain \( 0 \leq n-k \leq n \). Therefore, we can split the product \( 1 \cdot 2 \cdot \ldots \cdot n \) into two smaller products by putting its first \( n-k \) factors into the first block and its last \( k \) factors into the second:
\[
1 \cdot 2 \cdot \ldots \cdot n = (1 \cdot 2 \cdot \ldots \cdot (n-k)) \cdot ((n-k+1) \cdot (n-k+2) \cdot \ldots \cdot n).
\]

Now, the definition of \( n! \) yields
\[
n! = 1 \cdot 2 \cdot \ldots \cdot n
= \underbrace{(1 \cdot 2 \cdot \ldots \cdot (n-k))}_{=(n-k)!} \cdot \underbrace{((n-k+1) \cdot (n-k+2) \cdot \ldots \cdot n)}_{=n(n-1)(n-2)\cdots(n-k+1)}
\]
(since \( (n-k)! \) was defined as \( 1 \cdot 2 \cdot \ldots \cdot (n-k) \)) (here, we have reversed the order of multiplication)
\[
= (n-k)! \cdot (n(n-1)(n-2)\cdots(n-k+1)).
\]
Solving this for \( n \, (n - 1) \, (n - 2) \cdots (n - k + 1) \), we obtain
\[
n \, (n - 1) \, (n - 2) \cdots (n - k + 1) = n! \, / \, (n - k)!.
\] (18)

Now, \( k \in \mathbb{N} \); thus, the definition of \( \binom{n}{k} \) yields
\[
\binom{n}{k} = \frac{n \, (n - 1) \, (n - 2) \cdots (n - k + 1)}{k!} = \frac{n!}{k!} \quad \text{(by (18))}
\]
\[
= \frac{n!}{k! \, (n - k)!}.
\]

This solves part (a) of the exercise.

**b** Let \( n \in \mathbb{N} \) and \( k \in \mathbb{Q} \) be such that \( k > n \). We must prove that \( \binom{n}{k} = 0 \).

If \( k \not\in \mathbb{N} \), then this follows immediately from the definition of \( \binom{n}{k} \) (since \( \binom{n}{k} \) is simply defined to be 0 in this case). Thus, we WLOG assume that \( k \in \mathbb{N} \) for the rest of this proof.

From \( k > n \), we obtain \( n < k \), thus \( n \leq k - 1 \) (since both \( n \) and \( k \) are integers\(^3\)). Thus, one of the \( k \) factors of the product \( \prod_{i=0}^{k-1} (n - i) \) is \( n - n = 0 \). Therefore, this product \( \prod_{i=0}^{k-1} (n - i) \) has at least one factor equal to 0; thus, the whole product is 0. In other words, \( \prod_{i=0}^{k-1} (n - i) = 0 \). Now, the definition of \( \binom{n}{k} \) yields
\[
\binom{n}{k} = \frac{\prod_{i=0}^{k-1} (n - i)}{k!} = \frac{0}{k!}
\]
(since \( \prod_{i=0}^{k-1} (n - i) = 0 \)). Thus, \( \binom{n}{k} = \frac{0}{k!} = 0 \). This solves part (b) of the exercise.

**c** Let \( n \in \mathbb{N} \) and \( k \in \mathbb{Q} \). We must prove the equality (14). If \( k \) is not an integer, then this equality trivially holds.\(^4\) Hence, for the rest of this proof, we WLOG assume that \( k \) is an integer.

We are in one of the following three cases:

Case 1: We have \( k < 0 \).

Case 2: We have \( k > n \).

Case 3: We have neither \( k < 0 \) nor \( k > n \).

\(^3\)thanks to the \( k \in \mathbb{N} \) assumption that we just made

\(^4\)Proof. Assume that \( k \) is not an integer. If \( n - k \) was an integer, then \( k = n - (n - k) \) would be an integer as well (being the difference of the two integers \( n \) and \( n - k \)), which would contradict the fact that \( k \) is not an integer. Hence, \( n - k \) cannot be an integer. Thus, \( n - k \not\in \mathbb{N} \). Hence, \( \binom{n}{n-k} = 0 \) (by the definition of \( \binom{n}{n-k} \)). Also, \( k \not\in \mathbb{N} \) (since \( k \) is not an integer); thus, \( \binom{n}{k} = 0 \) (by the definition of \( \binom{n}{k} \)). Comparing these two equalities, we obtain \( \binom{n}{k} = \binom{n}{n-k} \). In other words, (14) holds. Thus, we have proven (14) in the case when \( k \) is not an integer.
Let us first consider Case 1. In this case, we have $k < 0$. Thus, $k \notin \mathbb{N}$, so that \( \binom{n}{k} = 0 \) (by the definition of \( \binom{n}{k} \)). On the other hand, from $k < 0$, we obtain $n - k > n$. Hence, part (b) of this exercise (applied to $n - k$ instead of $k$) yields \( \binom{n}{n-k} = 0 \). Comparing this with \( \binom{n}{k} = 0 \), we obtain \( \binom{n}{k} = \binom{n}{n-k} \). Hence, (14) is proven in Case 1.

Let us next consider Case 2. In this case, we have $k > n$. Thus, $n - k < 0$, so that $n - k \notin \mathbb{N}$, and thus \( \binom{n}{n-k} = 0 \) (by the definition of \( \binom{n}{n-k} \)). On the other hand, part (b) of this exercise yields \( \binom{n}{k} = 0 \). Comparing this with \( \binom{n}{n-k} = 0 \), we obtain \( \binom{n}{k} = \binom{n}{n-k} \). Hence, (14) is proven in Case 2.

Let us finally consider Case 3. In this case, we have neither $k < 0$ nor $k > n$. Hence, we have $k \geq 0$ and $k \leq n$. Thus, $n \geq k$ and $k \in \mathbb{N}$ (since $k \geq 0$). Hence, part (a) of this exercise yields \( \binom{n}{k} = \frac{n!}{k! (n-k)!} \). Also, $n - k \geq 0$ (since $n \geq k$), so that $n - k \in \mathbb{N}$. Also, from $k \geq 0$, we get $n \geq n - k$. Thus, part (a) of this exercise (applied to $n - k$ instead of $k$) yields
\[
\binom{n}{n-k} = \frac{n!}{(n-k)! (n-(n-k))!} = \frac{n!}{(n-k)! k! (n-k)!}.
\]
Comparing this with \( \binom{n}{k} = \frac{n!}{k! (n-k)!} \), we obtain \( \binom{n}{k} = \binom{n}{n-k} \). Hence, (14) is proven in Case 3.

We have now proven (14) in all three Cases 1, 2 and 3. Thus, (14) always holds. This solves part (c) of the exercise.

(d) Let $n \in \mathbb{Q}$ and $k \in \mathbb{Q}$. We must prove the equality (15). If $k \notin \mathbb{N}$, then this equality trivially holds.

Hence, for the rest of this proof, we WLOG assume that $k \in \mathbb{N}$.

Thus, the definition of \( \binom{-n}{k} \) yields
\[
\binom{-n}{k} = \frac{(-n) \cdot ((-n) - 1) \cdot ((-n) - 2) \cdots ((-n) - k + 1)}{k!}
= \frac{1}{k!} \cdot \frac{(-n) \cdot ((-n) - 1) \cdot ((-n) - 2) \cdots ((-n) - k + 1)}{(-n) \cdot (-n+1) \cdot \cdots \cdot (-n+k-1)}
= (-1)^k \cdot \frac{n \cdot (n+1) \cdot (n+2) \cdots (n+k-1)}{n \cdot (n+1) \cdot (n+2) \cdots (n+k-1)}
= (-1)^k \cdot \frac{n \cdot (n+1) \cdot (n+2) \cdots (n+k-1)}{n \cdot (n+1) \cdot (n+2) \cdots (n+k-1)}.
\]

(19)

Proof. Assume that $k \notin \mathbb{N}$. Then, \( \binom{-n}{k} = 0 \) (by the definition of \( \binom{-n}{k} \)) and \( \binom{k+n-1}{k} = 0 \) (by the definition of \( \binom{k+n-1}{k} \)). In view of these two equations, the equality (15) rewrites as $0 = (-1)^k 0$, which is obviously true. Thus, we have proven (15) in the case when $k \notin \mathbb{N}$.
On the other hand, the definition of \( \binom{k + n - 1}{k} \) yields

\[
\binom{k + n - 1}{k} = \frac{(k + n - 1)((k + n - 1) - 1)((k + n - 1) - 2) \cdots ((k + n - 1) - k + 1)}{k!}
\]

\[
= \frac{1}{k!} \frac{(k + n - 1)((k + n - 1) - 1)((k + n - 1) - 2) \cdots ((k + n - 1) - k + 1)}{(k + n - 1)(k + n - 2)(k + n - 3) \cdots n} = n(n + 1)(n + 2) \cdots (n + k - 1)
\]

(here, we have reversed the order of multiplication)

\[
= \frac{1}{k!} (n(n + 1)(n + 2) \cdots (n + k - 1)),
\]

so that

\[
(-1)^k \binom{k + n - 1}{k} = (-1)^k \frac{1}{k!} (n(n + 1)(n + 2) \cdots (n + k - 1)) = \frac{1}{k!} (-1)^k (n(n + 1)(n + 2) \cdots (n + k - 1)).
\]

Comparing this with (19), we obtain \( \binom{-n}{k} = (-1)^k \binom{k + n - 1}{k} \). Thus, (15) is proven. This solves part (d) of the exercise.

(e) Let \( n \in \mathbb{Q} \) and \( k \in \mathbb{Q} \). We must prove the equality (16). If \( k \notin \mathbb{N} \), then this equality trivially holds. Hence, for the rest of this proof, we WLOG assume that \( k \in \mathbb{N} \).

We are in one of the following two cases:

**Case 1:** We have \( k = 0 \).

**Case 2:** We have \( k \neq 0 \).

Let us first consider Case 1. In this case, we have \( k = 0 \). Thus, \( k - 1 = -1 \notin \mathbb{N} \), so that

\[
\binom{n - 1}{k - 1} = 0 \quad \text{(by the definition of} \quad \binom{n - 1}{k - 1}).
\]

But \( 0 \in \mathbb{N} \); thus, the definition of \( \binom{n}{0} \) yields

\[
\frac{n}{0!} = \frac{n(n - 1)(n - 2) \cdots (n - 0 + 1)}{0!}.
\]

Since \( n(n - 1)(n - 2) \cdots (n - 0 + 1) = (\text{empty product}) = 1 \) and \( 0! = 1 \), this rewrites as

\[
\binom{n}{0} = \frac{1}{1} = 1.
\]

This rewrites as \( \binom{n}{k} = 1 \) (since \( k = 0 \)). The same argument (applied to \( n - 1 \) instead of \( n \)) yields \( \binom{n - 1}{k} = 1 \). Now, the equality (16) boils down to \( 1 + 0 \) (since \( \binom{n}{k} = 1 \) and \( \binom{n - 1}{k} = 1 \) and \( \binom{n - 1}{k - 1} = 0 \)), which is true. Hence, (16) is proven in Case 1.

Proof. Assume that \( k \notin \mathbb{N} \). If we had \( k - 1 \in \mathbb{N} \), then we would have \( k = k - 1 + 1 \in \mathbb{N} \) as well, which would contradict the fact that \( k \notin \mathbb{N} \). Hence, we must have \( k - 1 \notin \mathbb{N} \). Hence, \( \binom{n - 1}{k - 1} = 0 \) (by the definition of \( \binom{n - 1}{k - 1} \)). Also, \( k \notin \mathbb{N} \); thus, \( \binom{n}{k} = 0 \) (by the definition of \( \binom{n}{k} \)) and \( \binom{n - 1}{k} = 0 \) (by the definition of \( \binom{n - 1}{k} \)). Now, the equality (16) boils down to \( 0 + 0 \) (since \( \binom{n}{k} = 0 \) and \( \binom{n - 1}{k} = 0 \) and \( \binom{n - 1}{k - 1} = 0 \)), which is clearly true. Thus, we have proven (16) in the case when \( k \notin \mathbb{N} \).
Let us first consider Case 2. In this case, we have \( k \neq 0 \). Thus, \( k \) is a positive integer (since \( k \in \mathbb{N} \)), so that \( k - 1 \in \mathbb{N} \).

Exercise 2 (a) (applied to \( k \) instead of \( n \)) yields \( k! = k \cdot (k - 1)! \), so that \( (k - 1)! = k!/k \) and thus

\[
\frac{1}{(k - 1)!} = \frac{1}{k!/k} = \frac{1}{k!} \cdot k.
\]

Recall that \( k - 1 \in \mathbb{N} \). Hence, the definition of \( \binom{n}{k - 1} \) yields

\[
\binom{n}{k - 1} = \frac{n(n - 1)(n - 2) \cdots (n - (k - 1) + 1)}{(k - 1)!} = \frac{1}{(k - 1)!} \cdot (n(n - 1)(n - 2) \cdots (n - (k - 1) + 1)).
\]

The same argument (applied to \( n - 1 \) instead of \( n \)) yields

\[
\binom{n - 1}{k - 1} = \frac{1}{(k - 1)!} \cdot \frac{(n - 1)((n - 1) - 1)((n - 1) - 2) \cdots ((n - 1) - (k - 1) + 1)}{(n - (k - 1)!)} = \frac{1}{k!} \cdot k \cdot ((n - 1)(n - 2) \cdots (n - k + 1)). \tag{20}
\]

On the other hand, the definition of \( \binom{n}{k} \) yields

\[
\binom{n}{k} = \frac{n(n - 1)(n - 2) \cdots (n - k + 1)}{k!} = \frac{1}{k!} (n(n - 1)(n - 2) \cdots (n - k + 1)). \tag{21}
\]

The same argument (applied to \( n - 1 \) instead of \( n \)) yields

\[
\binom{n - 1}{k} = \frac{1}{k!} \frac{(n - 1)((n - 1) - 1)((n - 1) - 2) \cdots ((n - 1) - k + 1)}{(n - k)!} = \frac{1}{k!} \cdot (n - k) \cdot ((n - 1)(n - 2) \cdots (n - k + 1)) \cdot (n - k)
\]

\[
= \frac{1}{k!} \cdot (n - k) \cdot ((n - 1)(n - 2) \cdots (n - k + 1)).
\]
Adding (20) to this equality, we obtain
\[
\binom{n-1}{k} + \binom{n-1}{k-1}
= \frac{1}{k!} (n-k) \cdot ((n-1)(n-2) \cdots (n-k+1))
+ \frac{1}{k!} \cdot k \cdot ((n-1)(n-2) \cdots (n-k+1))
= \frac{1}{k!} \cdot \frac{(n-k)+k}{(n-k)+k} \cdot ((n-1)(n-2) \cdots (n-k+1))
= \frac{1}{k!} \cdot n \cdot ((n-1)(n-2) \cdots (n-k+1))
= \frac{1}{k!} (n(n-1)(n-2) \cdots (n-k+1)) = \binom{n}{k}
\]
(by (21)). Hence, (16) is proven in Case 2.

We have now proven (16) in both Cases 1 and 2. Thus, (16) always holds. This solves part (e) of the exercise.

(f) Let \( n \in \mathbb{Q} \) and \( k \in \mathbb{Q} \). We must prove the equality (17). If \( k \not\in \mathbb{N} \), then this equality trivially holds. Hence, for the rest of this proof, we WLOG assume that \( k \in \mathbb{N} \).

We are in one of the following two cases:

Case 1: We have \( k = 0 \).

Case 2: We have \( k \neq 0 \).

Let us first consider Case 1. In this case, we have \( k = 0 \). Thus, \( k-1 = -1 \not\in \mathbb{N} \), so that
\[
\binom{n-1}{k-1} = 0 \quad \text{(by the definition of \( \binom{n-1}{k-1} \)).}
\]
Hence, \( n \binom{n-1}{k-1} = n \cdot 0 = 0 \). Comparing this with \( k \binom{n}{k} = 0 \), we obtain \( k \binom{n}{k} = n \binom{n-1}{k-1} \). Hence, (17) is proven in Case 1.

Let us first consider Case 2. In this case, we have \( k \neq 0 \). Thus, \( k \) is a positive integer (since \( k \in \mathbb{N} \)), so that \( k-1 \in \mathbb{N} \).

As in the solution to part (e) above, we can prove the equality (20). Multiplying both sides of this equality by \( n \), we obtain
\[
n \binom{n-1}{k-1} = n \cdot \frac{1}{k!} \cdot k \cdot ((n-1)(n-2) \cdots (n-k+1))
= k \cdot \frac{1}{k!} \cdot n \cdot ((n-1)(n-2) \cdots (n-k+1))
= k \cdot \frac{1}{k!} \cdot n(n-1)(n-2) \cdots (n-k+1).
\]

\(7\)Proof. Assume that \( k \not\in \mathbb{N} \). If we had \( k-1 \in \mathbb{N} \), then we would have \( k = k-1 + \frac{1}{k} \in \mathbb{N} \) as well, which would contradict the fact that \( k \not\in \mathbb{N} \). Hence, we must have \( k-1 \not\in \mathbb{N} \). Hence, \( \binom{n-1}{k-1} = 0 \) (by the definition of \( \binom{n-1}{k-1} \)). Also, \( k \not\in \mathbb{N} \); thus, \( \binom{n}{k} = 0 \) (by the definition of \( \binom{n}{k} \)). Now, the equality (17) boils down to \( k \cdot 0 = n \cdot 0 \) (since \( \binom{n}{k} = 0 \) and \( \binom{n-1}{k-1} = 0 \)), which is clearly true (since both sides equal 0). Thus, we have proven (17) in the case when \( k \not\in \mathbb{N} \).
On the other hand, the definition of \( \binom{n}{k} \) yields
\[
\binom{n}{k} = \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!} = \frac{1}{k!} \frac{n(n-1)(n-2)\cdots(n-k+1)}{n-k+1}.
\] (23)

Multiplying both sides of this equality by \( k \), we find
\[
k \binom{n}{k} = k \cdot \frac{1}{k!} \frac{n(n-1)(n-2)\cdots(n-k+1)}{n-k+1}.
\]

Comparing this with (22), we obtain \( k \binom{n}{k} = n \binom{n-1}{k-1} \). Hence, (17) is proven in Case 2.

We have now proven (17) in both Cases 1 and 2. Thus, (17) always holds. This solves part (f) of the exercise.

4 Exercise 4: General associativity for binary operations

4.1 Problem

[...]

4.2 Solution

[...]

REFERENCES


The numbering of theorems and formulas in this link might shift when the project gets updated; for a “frozen” version whose numbering is guaranteed to match that in the citations above, see https://github.com/darijgr/detnotes/releases/tag/2019-01-10.