1 Exercise 1: Mutual divisibility is rare

1.1 Problem
Let $a$ and $b$ be two integers such that $a \mid b$ and $b \mid a$. Prove that $|a| = |b|$.

1.2 Solution
[...]

2 Exercise 2: Congruence means equal remainders

2.1 Problem
Let $n$ be a positive integer. Let $u$ and $v$ be two integers. Prove that $u \equiv v \mod n$ if and only if $u\%n = v\%n$. 
2.2 Solution

[...]

3 Exercise 3: Even and Odd

3.1 Problem

Let $u$ be an integer.

(a) Prove that $u$ is even if and only if $u \% 2 = 0$.

(b) Prove that $u$ is odd if and only if $u \% 2 = 1$.

(c) Prove that $u$ is even if and only if $u \equiv 0 \mod 2$.

(d) Prove that $u$ is odd if and only if $u \equiv 1 \mod 2$.

(e) Prove that $u$ is odd if and only if $u + 1$ is even.

(f) Prove that exactly one of the two numbers $u$ and $u + 1$ is even.

(g) Prove that $u (u + 1) \equiv 0 \mod 2$.

(h) Prove that $u^2 \equiv -u \equiv u \mod 2$.

3.2 Solution

[...]

4 Exercise 4: Factorials 102

4.1 Problem

(a) Prove that

$$\frac{1! \cdot 2! \cdots (2n)!}{n!} = 2^n \cdot \prod_{i=1}^{n} \frac{(2i - 1)!}{((2i - 1)!)^2}$$

for each $n \in \mathbb{N}$.

(b) Prove that

$$\sum_{k=0}^{n} \frac{1}{k! \cdot (k + 2)} = 1 - \frac{1}{(n + 2)!}$$

for each $n \in \mathbb{N}$.
4.2 Solution

[...]

5 Exercise 5: Binomial coefficients

5.1 Problem

Prove that

$$\frac{(ab)!}{a!(bl)^a} = \prod_{k=1}^{a} \left( \frac{kb - 1}{b - 1} \right)$$

for all $a \in \mathbb{N}$ and all positive integers $b$.

5.2 Solution

[...]

6 Exercise 6: Binomial coefficients and coprimality

6.1 Problem

It is well-known (see, e.g., [Grinbe19, Proposition 3.20]) that $\binom{n}{k} \in \mathbb{Z}$ for all $n \in \mathbb{Z}$ and $k \in \mathbb{N}$. (This is not at all clear from the definition of $\binom{n}{k}$; it is saying that the product of any $k$ consecutive integers is divisible by $k!$. The case of $k = 2$ is the statement of Exercise 3 (g).) Thus, we can study the divisibility of binomial coefficients by various integers. There are hundreds of theorems about this; this exercise is about one of them.

Let $a$ and $b$ be two coprime positive integers.

(a) Prove that $\frac{a}{a+b} \left( \begin{array}{c} a+b \\ a \end{array} \right) = \left( \begin{array}{c} a+b-1 \\ a-1 \end{array} \right)$ and $\frac{b}{a+b} \left( \begin{array}{c} a+b \\ a \end{array} \right) = \left( \begin{array}{c} a+b-1 \\ b-1 \end{array} \right)$.

(b) Prove that if $h \in \mathbb{Q}$ satisfies $ah \in \mathbb{Z}$ and $bh \in \mathbb{Z}$, then $h \in \mathbb{Z}$. (This is where the coprimality of $a$ and $b$ comes into play.)

(c) Prove that $a+b \mid \left( \begin{array}{c} a+b \\ a \end{array} \right)$.

(d) Find a counterexample to the claim of part (c) if $a$ and $b$ are allowed to not be coprime.

6.2 Solution

[...]
REFERENCES

See [https://www-cs-faculty.stanford.edu/~knuth/gkp.html](https://www-cs-faculty.stanford.edu/~knuth/gkp.html) for errata.

The numbering of theorems and formulas in this link might shift when the project gets updated; for a “frozen” version whose numbering is guaranteed to match that in the citations above, see [https://github.com/darijgr/detnotes/releases/tag/2019-01-10](https://github.com/darijgr/detnotes/releases/tag/2019-01-10).

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