1 Exercise 1: Mutual divisibility is rare

1.1 Problem
Let $a$ and $b$ be two integers such that $a \mid b$ and $b \mid a$. Prove that $|a| = |b|$.

1.2 Solution
See the class notes, where this is Exercise 2.2.2. (The numbering may shift; it is one of the exercises in the "Divisibility" section.)

2 Exercise 2: Congruence means equal remainders

2.1 Problem
Let $n$ be a positive integer. Let $u$ and $v$ be two integers. Prove that $u \equiv v \mod n$ if and only if $u \% n = v \% n$. 

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2.2 Solution

See the class notes, where this is Exercise 2.6.1. (The numbering may shift; it is one of the exercises in the “Division with remainder” section.)

3 Exercise 3: Even and Odd

3.1 Problem

Let \( u \) be an integer.

(a) Prove that \( u \) is even if and only if \( u \equiv 0 \pmod{2} \).

(b) Prove that \( u \) is odd if and only if \( u \equiv 1 \pmod{2} \).

(c) Prove that \( u \) is even if and only if \( u \equiv 0 \pmod{2} \).

(d) Prove that \( u \) is odd if and only if \( u \equiv 1 \pmod{2} \).

(e) Prove that \( u \) is odd if and only if \( u + 1 \) is even.

(f) Prove that exactly one of the two numbers \( u \) and \( u + 1 \) is even.

(g) Prove that \( u (u + 1) \equiv 0 \pmod{2} \).

(h) Prove that \( u^2 \equiv -u \equiv u \pmod{2} \).

3.2 Solution

See the class notes, where this is Exercise 2.7.1 parts (a) to (h). (The numbering may shift; it is one of the exercises in the “Even and odd numbers” section.)

4 Exercise 4: Factorials 102

4.1 Problem

(a) Prove that

\[
\frac{1! \cdot 2! \cdot \ldots \cdot (2n)!}{n!} = 2^n \cdot \prod_{i=1}^{n} ((2i - 1)!)^2 \quad \text{for each } n \in \mathbb{N}.
\]

(b) Prove that

\[
\sum_{k=0}^{n} \frac{1}{k! \cdot (k + 2)} = 1 - \frac{1}{(n + 2)!} \quad \text{for each } n \in \mathbb{N}.
\]
4.2 Solution

We first recall that

\[ n! = n \cdot (n-1)! \quad \text{for each positive integer } n. \]  

(This was the claim of Exercise 2 (a) on homework set #0.)

(a) This is precisely [Grinbe19, Exercise 3.5 (c)], with only a superficial difference (namely, I write “\( \prod_{i=1}^{n} \frac{((2i-1)!)^2}{(2i)!} \)” instead of “\( \prod_{i=1}^{n} (2i-1)!^{2n} \)” in [Grinbe19, Exercise 3.5 (c)], but these two expressions are clearly equivalent). I give two solutions in [Grinbe19, solution to Exercise 3.5 (c)]: one by manipulation and one by induction. Here I will only show the solution by manipulation:

Let \( n \in \mathbb{N} \). Then, we can group the factors of the product \( 1! \cdot 2! \cdot \cdots \cdot (2n)! \) into pairs of successive factors. We thus obtain

\[
1! \cdot 2! \cdot \cdots \cdot (2n)! = (1! \cdot 2!) \cdot (3! \cdot 4!) \cdot \cdots \cdot ((2n-1)! \cdot (2n)!) = \prod_{i=1}^{n} \frac{((2i-1)!)^2}{(2i)!} = \left( \prod_{i=1}^{n} \frac{(2i)!}{(2i-1)!} \right) = \prod_{i=1}^{n} \frac{(2i)!}{((2i-1)!)^2}
\]

\[
= \prod_{i=1}^{n} (2i) \cdot \prod_{i=1}^{n} (2i-1)! = 2^n \prod_{i=1}^{n} \left( \frac{(2i)!}{(2i-1)!} \right) = 2^n \prod_{i=1}^{n} ((2i-1)!)^2.
\]

Dividing both sides of this equality by \( n! \), we find

\[
\frac{1! \cdot 2! \cdot \cdots \cdot (2n)!}{n!} = 2^n \prod_{i=1}^{n} ((2i-1)!)^2.
\]

This solves part (a) of the exercise.

(b) Again, the exercise can be proven by induction or by the telescope principle. Let me show the latter solution. First, I quote the telescope principle:

**Proposition 4.1.** Let \( m \in \mathbb{N} \). Let \( a_0, a_1, \ldots, a_m \) be \( m + 1 \) real numbers. Then,

\[
\sum_{i=1}^{m} (a_i - a_{i-1}) = a_m - a_0.
\]

Now, let me solve the exercise. Let \( n \in \mathbb{N} \). For each \( i \in \{0, 1, \ldots, n\} \), we set \( a_i = \frac{-1}{(i + 2)!} \).

Thus, \( a_0, a_1, \ldots, a_n \) are \( n + 1 \) real numbers. We state the following:

---

1Strictly speaking, we are tacitly using the fact that each integer between 1 and 2n (inclusive) can be written either in the form 2i or in the form 2i − 1 for some \( i \in \{1, 2, \ldots, n\} \), and that this \( i \) is unique. The proof of this fact relies on division with remainder.
Claim 1: For each $i \in \{0, 1, \ldots, n\}$, we have

$$a_i - a_{i-1} = \frac{1}{i! \cdot (i + 2)}.$$

[Proof of Claim 1: Let $i \in \{0, 1, \ldots, n\}$. Then, \(1\) (applied to $i + 1$ instead of $n$) yields $(i + 1)! = (i + 1) \cdot i!$. Also, \(1\) (applied to $i + 2$ instead of $n$) yields $(i + 2)! = (i + 2) \cdot (i + 1)!$.

The definition of $a_i$ yields

$$a_i = \frac{-1}{(i + 2)!} = \frac{-1}{(i + 2) \cdot (i + 1)!} \quad \text{(since $(i + 2)! = (i + 2) \cdot (i + 1)!$)}.$$

The definition of $a_{i-1}$ yields

$$a_{i-1} = \frac{-1}{(i - 1 + 2)!} = \frac{-1}{(i + 1)!} \quad \text{(since $(i - 1 + 2) = i + 1$)}.$$

Subtracting this equality from the previous one, we obtain

$$a_i - a_{i-1} = \frac{-1}{(i + 2) \cdot (i + 1)!} - \frac{-1}{(i + 1)!} = \frac{1}{(i + 1)!} - \frac{1}{(i + 2) \cdot (i + 1)!} = \frac{1}{(i + 2) \cdot (i + 1)!} \quad \text{(since $(i + 1)! = (i + 1) \cdot i!$)}.$$

This proves Claim 1.]

Now, Proposition 4.1 (applied to $m = n$) yields

$$\sum_{i=1}^{n} (a_i - a_{i-1}) = \sum_{i=1}^{n} a_i - a_0 = \frac{a_n}{(n + 2)!} - \frac{a_0}{(0 + 2)!} \quad \text{(by the definition of $a_n$)} \quad \text{(by the definition of $a_0$)}$$

$$= \frac{-1}{(n + 2)!} - \frac{-1}{(0 + 2)!} = \frac{1}{(0 + 2)!} - \frac{1}{(n + 2)!} = \frac{1 - \frac{1}{2}}{2!} = \frac{1}{2} - \frac{1}{(n + 2)!}.$$

Comparing this with

$$\sum_{i=1}^{n} \frac{1}{i! \cdot (i + 2)} = \sum_{i=1}^{n} \frac{1}{i! \cdot (i + 2)},$$

we obtain

$$\sum_{i=1}^{n} \frac{1}{i! \cdot (i + 2)} = \frac{1}{2} - \frac{1}{(n + 2)!}. \quad (2)$$
But
\[
\sum_{k=0}^{n} \frac{1}{k! \cdot (k + 2)} = \sum_{i=0}^{n} \frac{1}{i! \cdot (i + 2)} \quad \text{(here, we have renamed the summation index } k \text{ as } i)
\]
\[
= \frac{1}{0! \cdot (0 + 2)} + \sum_{i=1}^{n} \frac{1}{i! \cdot (i + 2)} = \frac{1}{2} + \frac{1}{(n + 2)!} = 1 - \frac{1}{(n + 2)!}.
\]
\[
\text{(by \ref{L3.10})}
\]
This solves part (b) of the exercise.

5 Exercise 5: Binomial Coefficients 102

5.1 Problem

Prove that
\[
\frac{(ab)!}{a! \cdot (b)!} = \prod_{k=1}^{a} \left( \frac{kb - 1}{b - 1} \right)
\]
for all \(a \in \mathbb{N}\) and all positive integers \(b\).

5.2 Solution

First, let us state an analogue of the telescope principle (Proposition \ref{prop-tv}) for products instead of sums:

Proposition 5.1. Let \(m \in \mathbb{N}\). Let \(a_0, a_1, \ldots, a_m\) be \(m + 1\) nonzero real numbers. Then,
\[
\prod_{i=1}^{m} \frac{a_i}{a_{i-1}} = \frac{a_m}{a_0}.
\]

Proof of Proposition 5.1. Take your favorite proof of Proposition \ref{prop-tv} and replace addition by multiplication, subtraction by division and sums by products. This will yield a proof of Proposition 5.1.

Furthermore, recall the following facts:

Proposition 5.2. If \(n \in \mathbb{N}\) and \(k \in \mathbb{N}\) are such that \(n \geq k\), then
\[
\binom{n}{k} = \frac{n!}{k! \cdot (n-k)!}.
\]

Proposition 5.2 is Exercise 3 (a) on homework set \#0

Proposition 5.3. Any \(n \in \mathbb{Q}\) and \(k \in \mathbb{Q}\) satisfy
\[
k \binom{n}{k} = n \binom{n-1}{k-1}.
\]
Proposition 5.3 is Exercise 3 (f) on homework set #0.

Now, let \(a \in \mathbb{N}\), and let \(b\) be a positive integer. Thus, \(b \neq 0\) (since \(b\) is positive).

Claim 1: We have
\[
\binom{kb - 1}{b - 1} = \frac{1}{k} \cdot \frac{1}{b!} \cdot \frac{(kb)!}{((k - 1)b)!}
\]
for each positive integer \(k\).

[Proof of Claim 1: Let \(k\) be a positive integer. Then, Proposition 5.3 (applied to \(kb\) and \(b\) instead of \(n\) and \(k\)) yields
\[
b\binom{kb}{b} = kb\binom{kb - 1}{k - 1}.
\]
We can cancel \(b\) from this equality (since \(b\) is nonzero), and thus obtain
\[
\binom{kb}{b} = k\binom{kb - 1}{k - 1}.
\]
On the other hand, \(k \geq 1\) (since \(k\) is a positive integer). We can multiply this inequality by \(b\) (since \(b\) is positive) and thus obtain \(kb \geq 1b = b\). Hence, Proposition 5.2 (applied to \(kb\) and \(b\) instead of \(n\) and \(k\)) yields
\[
\binom{kb}{b} = \frac{(kb)!}{b!(kb - b)!} = \frac{(kb)!}{b!((k - 1)b)!} \quad \text{(since } kb - b = (k - 1)b\text{)}.
\]
Comparing this equality with \(\binom{kb}{b} = k\binom{kb - 1}{k - 1}\), we obtain
\[
k\binom{kb - 1}{k - 1} = \frac{(kb)!}{b!((k - 1)b)!}.
\]
We can divide both sides of this equality by \(k\) (since \(k\) is positive and thus nonzero); thus we obtain
\[
\binom{kb - 1}{b - 1} = \frac{1}{k} \cdot \frac{(kb)!}{b!((k - 1)b)!} = \frac{1}{k} \cdot \frac{1}{b!} \cdot \frac{(kb)!}{((k - 1)b)!}.
\]
This proves Claim 1.]
Now,
\[
\prod_{k=1}^{a} \left( \frac{(kb - 1)}{(b - 1)} \right) = \frac{1}{k} \cdot \frac{1}{b!} \cdot \frac{(kb)!}{((k - 1)b)!} \quad \text{(by Claim 1)}
\]
\[
= \prod_{k=1}^{a} \left( \frac{1}{k} \cdot \frac{1}{b!} \cdot \frac{(kb)!}{((k - 1)b)!} \right) = \left( \prod_{k=1}^{a} \frac{1}{k} \right) \cdot \left( \prod_{k=1}^{a} \frac{1}{b!} \right) \cdot \left( \prod_{k=1}^{a} \frac{(kb)!}{((k - 1)b)!} \right)
\]
\[
= \frac{1}{a!} \cdot \frac{1}{(b!)^a} \cdot \prod_{i=1}^{a} \frac{(ib)!}{((i - 1)b)!} = \frac{1}{a!} \cdot \frac{1}{(b!)^a} \cdot \frac{(ab)!}{(0b)!} = \frac{(ab)!}{(0b)!} (\text{by Proposition 5.1, applied to } m = a \text{ and } a_i = (ib)!)
\]
\[
= \frac{1}{a!} \cdot \frac{1}{(b!)^a} \cdot \frac{(ab)!}{a! (b!)^a} = \frac{(ab)!}{a! (b!)^a}.
\]
This solves the exercise.

5.3 Remark

Proposition 2.17.12 in the class notes says that \( \binom{n}{k} \) is an integer for all \( n \in \mathbb{Z} \) and \( k \in \mathbb{Q} \). In other words,
\[
\binom{n}{k} \in \mathbb{Z} \quad \text{for all } n \in \mathbb{Z} \text{ and } k \in \mathbb{Q}.
\]
(3)

Using this fact and the above exercise, we can show the following:

Corollary 5.4. Let \( a \in \mathbb{N} \). Let \( b \) be a positive integer. Then, \( a! (b!)^a \mid (ab)! \).

Proof of Corollary 5.4. The exercise yields
\[
\frac{(ab)!}{a! (b!)^a} = \prod_{k=1}^{a} \frac{(kb - 1)}{(b - 1)} \in \mathbb{Z}.
\]
(by \( \text{Eq. 3} \), applied to \( kb - 1 \) and \( b - 1 \) instead of \( n \) and \( k \))

In other words, \( a! (b!)^a \mid (ab)! \). This proves Corollary 5.4.

We refer to [GrKnPa94, Chapter 5] for further properties of binomial coefficients.
6 Exercise 6: Binomial Coefficients and Coprimality

6.1 Problem

It is well-known (see, e.g., [Grinbe19, Proposition 3.20]) that
\[ \binom{n}{k} \in \mathbb{Z} \text{ for all } n \in \mathbb{Z} \text{ and } k \in \mathbb{N}. \]  
(This is not at all clear from the definition of \( \binom{n}{k} \); it is saying that the product of any \( k \) consecutive integers is divisible by \( k! \). The case of \( k = 2 \) is the statement of Exercise 3 (g).) Thus, we can study the divisibility of binomial coefficients by various integers. There are hundreds of theorems about this; this exercise is about one of them.

Let \( a \) and \( b \) be two coprime positive integers.

(a) Prove that \( \frac{a}{a+b} \binom{a+b}{a} = \binom{a+b-1}{a-1} \) and \( \frac{b}{a+b} \binom{a+b}{a} = \binom{a+b-1}{b-1} \).

(b) Prove that if \( h \in \mathbb{Q} \) satisfies \( ah \in \mathbb{Z} \) and \( bh \in \mathbb{Z} \), then \( h \in \mathbb{Z} \). (This is where the coprimality of \( a \) and \( b \) comes into play.)

(c) Prove that \( a+b \mid \binom{a+b}{a} \).

(d) Find a counterexample to the claim of part (c) if \( a \) and \( b \) are allowed to not be coprime.

6.2 Solution

We recall Bezout’s theorem (proven in the class notes):

**Theorem 6.1.** Let \( a \) and \( b \) be two integers. Then, there exist integers \( x \in \mathbb{Z} \) and \( y \in \mathbb{Z} \) such that
\[ \gcd(a, b) = xa + yb. \]

Both \( a \) and \( b \) are positive integers. Hence, the sum \( a+b \) is a positive integer as well. Thus, in particular, \( a+b \) is nonzero.

(a) We have \( a+b \geq a \) (since \( b \) is positive and thus nonnegative). Hence, Proposition 5.2 (applied to \( n = a+b \) and \( k = a \)) yields
\[ \binom{a+b}{a} = \frac{(a+b)!}{a!((a+b)-a)!} = \frac{(a+b)!}{a!b!} \quad \text{(since } (a+b) - a = b \text{).} \]

The same argument (but with the roles of \( a \) and \( b \) interchanged) yields
\[ \binom{b+a}{b} = \frac{(b+a)!}{b!a!} = \frac{(b+a)!}{a!b!} = \frac{(a+b)!}{a!b!} \quad \text{(since } b+a = a+b \text{).} \]

Comparing these two equalities, we obtain
\[ \binom{b+a}{b} = \binom{a+b}{a}. \]
Proposition 5.3 (applied to \( n = a + b \) and \( k = a \)) yields \( a \left( \frac{a + b}{a} \right) = (a + b) \left( \frac{a + b - 1}{a - 1} \right) \).

We can divide both sides of this equality by \( a + b \) (since \( a + b \) is nonzero). We thus obtain

\[
\frac{a}{a + b} \left( \frac{a + b}{a} \right) = \left( \frac{a + b - 1}{a - 1} \right).
\]

The same argument (but with the roles of \( a \) and \( b \) interchanged) yields

\[
\frac{b}{b + a} \left( \frac{b + a}{b} \right) = \left( \frac{b + a - 1}{b - 1} \right).
\]

In view of (5), this rewrites as

\[
\frac{b}{b + a} \left( \frac{a + b}{a} \right) = \left( \frac{b + a - 1}{b - 1} \right).
\]

In view of \( b + a = a + b \), this rewrites as

\[
\frac{b}{a + b} \left( \frac{a + b}{b} \right) = \left( \frac{a + b - 1}{b - 1} \right).
\]

Having proven both (6) and (7), we have thus solved part (a) of the exercise.

(b) Let \( h \in \mathbb{Q} \) satisfy \( ah \in \mathbb{Z} \) and \( bh \in \mathbb{Z} \). We must prove that \( h \in \mathbb{Z} \).

Theorem 6.1 shows that there exist integers \( x \in \mathbb{Z} \) and \( y \in \mathbb{Z} \) such that \( \gcd(a, b) = xa + yb \). Consider these \( x \) and \( y \).

But we know that \( a \) and \( b \) are coprime. In other words, \( \gcd(a, b) = 1 \). Hence, \( 1 = \gcd(a, b) = xa + yb \). Multiplying both sides of this equality by \( h \), we find

\[
h \cdot 1 = h(xa + yb) = x \cdot (ah) + y \cdot (bh) .
\]

All four numbers \( x, ah, y \) and \( bh \) on the right hand side of this equality are integers (since \( x \in \mathbb{Z} \), \( ah \in \mathbb{Z} \), \( y \in \mathbb{Z} \) and \( bh \in \mathbb{Z} \)). Thus, the right hand side of this equality is an integer. Therefore, so is the left hand side. In other words, \( h \cdot 1 \in \mathbb{Z} \). In other words, \( h \in \mathbb{Z} \). This solves part (b) of the exercise.

(c) Recall that \( a \) is a positive integer; hence, \( a \in \mathbb{N} \) and \( a - 1 \in \mathbb{N} \). Also, \( b - 1 \in \mathbb{N} \) (since \( b \) is a positive integer). Now, (4) yields \( \left( \frac{a + b}{a} \right) \in \mathbb{Z} \) (since \( a + b \in \mathbb{Z} \) and \( a \in \mathbb{N} \)) and \( \left( \frac{a + b - 1}{a - 1} \right) \in \mathbb{Z} \) (since \( a + b - 1 \in \mathbb{Z} \) and \( a - 1 \in \mathbb{N} \)) and \( \left( \frac{a + b - 1}{b - 1} \right) \in \mathbb{Z} \) (since \( a + b - 1 \in \mathbb{Z} \) and \( b - 1 \in \mathbb{N} \)).

Define \( h \in \mathbb{Q} \) by \( h = \frac{1}{a + b} \left( \frac{a + b}{a} \right) \). (This is well-defined, since \( a + b \) is nonzero and \( \left( \frac{a + b}{a} \right) \) belongs to \( \mathbb{Z} \).)

From \( h = \frac{1}{a + b} \left( \frac{a + b}{a} \right) \), we obtain

\[
ah = a \cdot \frac{1}{a + b} \left( \frac{a + b}{a} \right) = \frac{a}{a + b} \left( \frac{a + b}{a} \right) = \left( \frac{a + b - 1}{a - 1} \right) \quad \text{(by part (a) of this exercise)}
\]
\( \in \mathbb{Z} \).
From \( h = \frac{1}{a + b} \binom{a + b}{a} \), we also obtain
\[
h \cdot \frac{1}{a + b} \binom{a + b}{a} = \frac{b}{a + b} \binom{a + b}{a} = \binom{a + b - 1}{b - 1}
\]
(by part (a) of this exercise)
\[
e \in \mathbb{Z}.
\]
Thus, part (b) of this exercise yields \( h \in \mathbb{Z} \). In view of
\[
h = \frac{1}{a + b} \binom{a + b}{a} = \frac{a}{a + b},
\]
this rewrites as \( \frac{a}{a + b} \in \mathbb{Z} \). In other words, \( a + b \mid \binom{a + b}{a} \) (since \( a + b \) is nonzero). This solves part (c) of the exercise.

(d) For example, setting \( a = 2 \) and \( b = 2 \) yields a counterexample, since \( 2 + 2 \nmid \binom{2 + 2}{2} \).
(In fact, \( 2 + 2 = 4 \nmid \binom{4}{2} = \binom{2 + 2}{2} \).)

REFERENCES

See [https://www-cs-faculty.stanford.edu/~knuth/gkp.html](https://www-cs-faculty.stanford.edu/~knuth/gkp.html) for errata.

The numbering of theorems and formulas in this link might shift when the project gets updated; for a “frozen” version whose numbering is guaranteed to match that in the citations above, see [https://github.com/darijgr/detnotes/releases/tag/2019-01-10](https://github.com/darijgr/detnotes/releases/tag/2019-01-10).