1 Exercise 1: Mutual divisibility is rare

1.1 Problem
Let $a$ and $b$ be two integers such that $a \mid b$ and $b \mid a$. Prove that $|a| = |b|$.

1.2 Solution
See the class notes, where this is Exercise 2.2.2. (The numbering may shift; it is one of the exercises in the “Divisibility” section.)

2 Exercise 2: Congruence means equal remainders

2.1 Problem
Let $n$ be a positive integer. Let $u$ and $v$ be two integers. Prove that $u \equiv v \mod n$ if and only if $u\%n = v\%n$. 
2.2 **Solution**

See the class notes, where this is Exercise 2.6.1. (The numbering may shift; it is one of the exercises in the “Division with remainder” section.)

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3 **Exercise 3: Even and Odd**

3.1 **Problem**

Let $u$ be an integer.

(a) Prove that $u$ is even if and only if $u \% 2 = 0$.

(b) Prove that $u$ is odd if and only if $u \% 2 = 1$.

(c) Prove that $u$ is even if and only if $u \equiv 0 \mod 2$.

(d) Prove that $u$ is odd if and only if $u \equiv 1 \mod 2$.

(e) Prove that $u$ is odd if and only if $u + 1$ is even.

(f) Prove that exactly one of the two numbers $u$ and $u + 1$ is even.

(g) Prove that $u (u + 1) \equiv 0 \mod 2$.

(h) Prove that $u^2 \equiv -u \equiv u \mod 2$.

3.2 **Solution**

See the class notes, where this is Exercise 2.7.1 parts (a) to (h). (The numbering may shift; it is one of the exercises in the “Even and odd numbers” section.)

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4 **Exercise 4: Factorials 102**

4.1 **Problem**

(a) Prove that

\[
\frac{1! \cdot 2! \cdots (2n)!}{n!} = 2^n \cdot \prod_{i=1}^{n} ((2i - 1)!)^2 \quad \text{for each } n \in \mathbb{N}.
\]

(b) Prove that

\[
\sum_{k=0}^{n} \frac{1}{k! \cdot (k + 2)} = 1 - \frac{1}{(n + 2)!} \quad \text{for each } n \in \mathbb{N}.
\]
4.2 Solution

We first recall that
\[ n! = n \cdot (n-1)! \quad \text{for each positive integer } n. \quad (1) \]
(This was the claim of Exercise 2 (a) on homework set #0)

(a) This is precisely [Grinbe19, Exercise 3.5 (c)], with only a superficial difference (namely, I write “\( \prod_{i=1}^{n} ((2i - 1)!)^2 \)” instead of “\( \prod_{i=1}^{n} ((2i - 1)!2^n \)” in [Grinbe19, Exercise 3.5 (c)], but these two expressions are clearly equivalent). I give two solutions in [Grinbe19, solution to Exercise 3.5 (c)]: one by manipulation and one by induction. Here I will only show the solution by manipulation:

Let \( n \in \mathbb{N} \). Then, we can group the factors of the product \( 1! \cdot 2! \cdots (2n)! \) into pairs of successive factors. We thus obtain

\[
1! \cdot 2! \cdots (2n)! = (1! \cdot 2!) \cdot (3! \cdot 4!) \cdots ((2n - 1)! \cdot (2n)!) = \prod_{i=1}^{n} \frac{(2i)!}{(2i - 1)!} \cdot \frac{(2i)!}{(2i - 1)!} = (\prod_{i=1}^{n} \frac{(2i)!}{(2i - 1)!})^2.
\]

Dividing both sides of this equality by \( n! \), we find

\[
\frac{1! \cdot 2! \cdots (2n)!}{n!} = 2^n \cdot \prod_{i=1}^{n} ((2i - 1)!)^2.
\]

This solves part (a) of the exercise.

(b) Again, the exercise can be proven by induction or by the telescope principle. Let me show the latter solution. First, I quote the telescope principle:

**Proposition 4.1.** Let \( m \in \mathbb{N} \). Let \( a_0, a_1, \ldots, a_m \) be \( m + 1 \) real numbers. Then,

\[
\sum_{i=1}^{m} (a_i - a_{i-1}) = a_m - a_0.
\]

Now, let me solve the exercise. Let \( n \in \mathbb{N} \). For each \( i \in \{0, 1, \ldots, n\} \), we set \( a_i = \frac{-1}{(i + 2)!} \).

Thus, \( a_0, a_1, \ldots, a_n \) are \( n + 1 \) real numbers. We state the following:

\[^{1}\text{Strictly speaking, we are tacitly using the fact that each integer between 1 and } 2n \text{ (inclusive) can be written either in the form } 2i \text{ or in the form } 2i - 1 \text{ for some } i \in \{1, 2, \ldots, n\}, \text{ and that this } i \text{ is unique. The proof of this fact relies on division with remainder.} \]
Claim 1: For each $i \in \{0, 1, \ldots, n\}$, we have

$$a_i - a_{i-1} = \frac{1}{i! \cdot (i + 2)}.$$ 

[Proof of Claim 1: Let $i \in \{0, 1, \ldots, n\}$. Then, (1) (applied to $i + 1$ instead of $n$) yields $(i + 1)! = (i + 1) \cdot i!$. Also, (1) (applied to $i + 2$ instead of $n$) yields $(i + 2)! = (i + 2) \cdot (i + 1)!$. The definition of $a_i$ yields

$$a_i = \frac{-1}{(i + 2)!} = \frac{-1}{(i + 2) \cdot (i + 1)!} \quad \text{(since } (i + 2)! = (i + 2) \cdot (i + 1)! \text{)}.$$ 

The definition of $a_{i-1}$ yields

$$a_{i-1} = \frac{-1}{(i - 1)!} = \frac{-1}{(i + 1)!} \quad \text{(since } (i - 1) + 2 = i + 1 \text{)}.$$ 

Subtracting this equality from the previous one, we obtain

$$a_i - a_{i-1} = \frac{-1}{(i + 2) \cdot (i + 1)!} - \frac{-1}{(i + 1)!} = \frac{1}{(i + 1)!} - \frac{1}{(i + 2) \cdot (i + 1)!} = \frac{(i + 2) - 1}{(i + 2) \cdot (i + 1)!} = \frac{i + 1}{(i + 2) \cdot (i + 1)!} \quad \text{(since } (i + 1)! = (i + 1) \cdot i! \text{)}.$$ 

This proves Claim 1.]

Now, Proposition 4.1 (applied to $m = n$) yields

$$\sum_{i=1}^{n} (a_i - a_{i-1}) = a_n - a_0 = \frac{-1}{(n + 2)!} - \frac{-1}{(0 + 2)!} \quad \text{(by the definition of } a_n, a_0 \text{)}$$

$$= \frac{-1}{(n + 2)!} - \frac{-1}{(0 + 2)!} = \frac{1}{(0 + 2)!} - \frac{1}{(n + 2)!} = \frac{1}{2} - \frac{1}{(n + 2)!}.$$ 

Comparing this with

$$\sum_{i=1}^{n} (a_i - a_{i-1}) = \sum_{i=1}^{n} \frac{1}{i! \cdot (i + 2)},$$

we obtain

$$\sum_{i=1}^{n} \frac{1}{i! \cdot (i + 2)} = \frac{1}{2} - \frac{1}{(n + 2)!}.$$ 

(2)
But
\[
\sum_{k=0}^{n} \frac{1}{k! \cdot (k + 2)} = \sum_{i=0}^{n} \frac{1}{i! \cdot (i + 2)} \quad \text{(here, we have renamed the summation index } k \text{ as } i)
\]
\[
= \frac{1}{0! \cdot (0 + 2)} + \sum_{i=1}^{n} \frac{1}{i! \cdot (i + 2)} = \frac{1}{2} + \frac{1}{2} = 1 - \frac{1}{(n + 2)!}.
\]

This solves part (b) of the exercise.

5 EXERCISE 5: BINOMIAL COEFFICIENTS 102

5.1 PROBLEM

Prove that
\[
\frac{(ab)!}{a! \cdot (b)!} = \prod_{k=1}^{a} \left( \frac{kb - 1}{b - 1} \right)
\]
for all \( a \in \mathbb{N} \) and all positive integers \( b \).

5.2 SOLUTION

First, let us state an analogue of the telescope principle (Proposition 4.1) for products instead of sums:

**Proposition 5.1.** Let \( m \in \mathbb{N} \). Let \( a_0, a_1, \ldots, a_m \) be \( m + 1 \) nonzero real numbers. Then,
\[
\prod_{i=1}^{m} \frac{a_i}{a_{i-1}} = \frac{a_m}{a_0}.
\]

**Proof of Proposition 5.1.** Take your favorite proof of Proposition 4.1 and replace addition by multiplication, subtraction by division and sums by products. This will yield a proof of Proposition 5.1.

Furthermore, recall the following facts:

**Proposition 5.2.** If \( n \in \mathbb{N} \) and \( k \in \mathbb{N} \) are such that \( n \geq k \), then
\[
\binom{n}{k} = \frac{n!}{k! (n - k)!}.
\]

Proposition 5.2 is Exercise 3 (a) on homework set #0.

**Proposition 5.3.** Any \( n \in \mathbb{Q} \) and \( k \in \mathbb{Q} \) satisfy
\[
k \binom{n}{k} = n \binom{n - 1}{k - 1}.
\]
Proposition 5.3 is Exercise 3 (f) on homework set \#0.

Now, let $a \in \mathbb{N}$, and let $b$ be a positive integer. Thus, $b \neq 0$ (since $b$ is positive).

Claim 1: We have

$${\begin{array}{*{20}c}
\binom{kb - 1}{b - 1} &= \frac{1}{k} \cdot \frac{1}{b!} \cdot \frac{(kb)!}{((k - 1)b)!} \\
\end{array}}$$

for each positive integer $k$.

Proof of Claim 1: Let $k$ be a positive integer. Then, Proposition 5.3 (applied to $kb$ and $b$ instead of $n$ and $k$) yields

$$b\binom{kb}{b} = kb\binom{kb - 1}{k - 1}.$$ 

We can cancel $b$ from this equality (since $b$ is nonzero), and thus obtain

$$\binom{kb}{b} = k\binom{kb - 1}{k - 1}.$$ 

On the other hand, $k \geq 1$ (since $k$ is a positive integer). We can multiply this inequality by $b$ (since $b$ is positive) and thus obtain $kb \geq 1b = b$. Hence, Proposition 5.2 (applied to $kb$ and $b$ instead of $n$ and $k$) yields

$$b\binom{kb}{b} = \frac{(kb)!}{b!(kb-b)!} = \frac{(kb)!}{b! ((k-1)b)!} \quad \text{(since $kb-b=(k-1)b$)}.$$ 

Comparing this equality with $\binom{kb}{b} = k\binom{kb-1}{k-1}$, we obtain

$$k\binom{kb-1}{k-1} = \frac{(kb)!}{b!((k-1)b)!}.$$ 

We can divide both sides of this equality by $k$ (since $k$ is positive and thus nonzero); thus we obtain

$$\binom{kb-1}{b-1} = \frac{1}{k} \cdot \frac{(kb)!}{b!((k-1)b)!} = \frac{1}{k} \cdot \frac{1}{b!} \cdot \frac{(kb)!}{((k-1)b)!}.$$ 

This proves Claim 1.
Now,
\[
\prod_{k=1}^{a} \left( \frac{kb - 1}{b - 1} \right)\]
\[= \frac{1}{k!} \cdot \frac{(kb)!}{((k-1)b)!} \quad \text{(by Claim 1)}\]
\[= \prod_{k=1}^{a} \left( \frac{1}{k!} \cdot \frac{1}{(k-1)b!} \right) = \prod_{k=1}^{a} \left( \frac{1}{k!} \cdot \frac{1}{(k-1)b!} \right) \cdot \prod_{k=1}^{a} \left( \frac{kb!}{(k-1)b!} \right)\]
\[= \prod_{k=1}^{a} \left( \frac{1}{k!} \cdot \frac{1}{(k-1)b!} \right) \cdot \prod_{k=1}^{a} \left( \frac{kb!}{(k-1)b!} \right) = \prod_{i=1}^{a} \left( \frac{(ib)!}{(i-1)b!} \right) \quad \text{applied to } m=a \text{ and } a_i=(ib)!\]
\[= \prod_{i=1}^{a} \left( \frac{(ib)!}{(i-1)b!} \right) = \frac{(ab)!}{(0b)!} = 1 \quad \text{(since } (0b)! = 0! = 1)\]
\[= \frac{1}{a!} \cdot \frac{1}{(bl)^a} \cdot \prod_{i=1}^{a} \left( \frac{(ib)!}{(i-1)b!} \right) \quad \text{by Proposition 5.1, applied to } m=a \text{ and } a_i=(ib)!\]
\[= \frac{1}{a!} \cdot \frac{1}{(bl)^a} \cdot \prod_{i=1}^{a} \left( \frac{(ib)!}{(i-1)b!} \right) = \frac{(ab)!}{(0b)!} = 1 \quad \text{since } (0b)! = 0! = 1\]
\[= \frac{1}{a!} \cdot \frac{1}{(bl)^a} \cdot \prod_{i=1}^{a} \left( \frac{(ib)!}{(i-1)b!} \right) = \frac{(ab)!}{a!(bl)^a}.\]

This solves the exercise.

6 Exercise 6: Binomial Coefficients and Coprimality

6.1 Problem

It is well-known (see, e.g., [Grinbe19 Proposition 3.20]) that

\[
\left( \binom{n}{k} \right) \in \mathbb{Z} \quad \text{for all } n \in \mathbb{Z} \text{ and } k \in \mathbb{N}. \quad (3)
\]

(This is not at all clear from the definition of \( \binom{n}{k} \); it is saying that the product of any \( k \) consecutive integers is divisible by \( k! \). The case of \( k = 2 \) is the statement of Exercise 3 (g).) Thus, we can study the divisibility of binomial coefficients by various integers. There are hundreds of theorems about this; this exercise is about one of them.

Let \( a \) and \( b \) be two coprime positive integers.

(a) Prove that \( \frac{a}{a+b} \binom{a+b}{a} = \binom{a+b-1}{a-1} \) and \( \frac{b}{a+b} \binom{a+b}{a} = \binom{a+b-1}{b-1} \).

(b) Prove that if \( h \in \mathbb{Q} \) satisfies \( ah \in \mathbb{Z} \) and \( bh \in \mathbb{Z} \), then \( h \in \mathbb{Z} \). (This is where the coprimality of \( a \) and \( b \) comes into play.)
(c) Prove that $a + b \mid \binom{a + b}{a}$. 

(d) Find a counterexample to the claim of part (c) if $a$ and $b$ are allowed to not be coprime.

6.2 SOLUTION

We recall Bezout’s theorem (proven in the class notes):

**Theorem 6.1.** Let $a$ and $b$ be two integers. Then, there exist integers $x \in \mathbb{Z}$ and $y \in \mathbb{Z}$ such that $\gcd(a, b) = xa + yb$.

Both $a$ and $b$ are positive integers. Hence, the sum $a + b$ is a positive integer as well. Thus, in particular, $a + b$ is nonzero.

(a) We have $a + b \geq a$ (since $b$ is positive and thus nonnegative). Hence, Proposition 5.2 (applied to $n = a + b$ and $k = a$) yields

$$
\binom{a + b}{a} = \frac{(a + b)!}{a!((a + b) - a)!} = \frac{(a + b)!}{a!b!} \quad (\text{since } (a + b) - a = b).
$$

The same argument (but with the roles of $a$ and $b$ interchanged) yields

$$
\binom{b + a}{b} = \frac{(b + a)!}{b!a!} = \frac{(b + a)!}{a!b!} = \frac{(a + b)!}{a!b!} \quad (\text{since } b + a = a + b).
$$

Comparing these two equalities, we obtain

$$
\binom{b + a}{b} = \binom{a + b}{a}. \quad (4)
$$

Proposition 5.3 (applied to $n = a + b$ and $k = a$) yields $a \binom{a + b}{a} = (a + b) \binom{a + b - 1}{a - 1}$. We can divide both sides of this equality by $a + b$ (since $a + b$ is nonzero). We thus obtain

$$
\frac{a}{a + b} \binom{a + b}{a} = \binom{a + b - 1}{a - 1}. \quad (5)
$$

The same argument (but with the roles of $a$ and $b$ interchanged) yields

$$
\frac{b}{b + a} \binom{b + a}{b} = \binom{b + a - 1}{b - 1}. \quad (6)
$$

In view of (4), this rewrites as

$$
\frac{b}{b + a} \binom{a + b}{a} = \binom{b + a - 1}{b - 1}. \quad (5)
$$

In view of $b + a = a + b$, this rewrites as

$$
\frac{b}{a + b} \binom{a + b}{b} = \binom{a + b - 1}{b - 1}. \quad (5)
$$

Having proven both (5) and (6), we have thus solved part (a) of the exercise.
(b) Let $h \in \mathbb{Q}$ satisfy $ah \in \mathbb{Z}$ and $bh \in \mathbb{Z}$. We must prove that $h \in \mathbb{Z}$.

Theorem 6.1 shows that there exist integers $x \in \mathbb{Z}$ and $y \in \mathbb{Z}$ such that $\gcd(a, b) = xa + yb$. Consider these $x$ and $y$.

But we know that $a$ and $b$ are coprime. In other words, $\gcd(a, b) = 1$. Hence, $1 = \gcd(a, b) = xa + yb$. Multiplying both sides of this equality by $h$, we find

$$h \cdot 1 = h(xa + yb) = x \cdot (ah) + y \cdot (bh).$$

All four numbers $x$, $ah$, $y$ and $bh$ on the right hand side of this equality are integers (since $x \in \mathbb{Z}$, $ah \in \mathbb{Z}$, $y \in \mathbb{Z}$ and $bh \in \mathbb{Z}$). Thus, the right hand side of this equality is an integer. Therefore, so is the left hand side. In other words, $h \cdot 1 \in \mathbb{Z}$. In other words, $h \in \mathbb{Z}$. This solves part (b) of the exercise.

(c) Recall that $a$ is a positive integer; hence, $a \in \mathbb{N}$ and $a - 1 \in \mathbb{N}$. Also, $b - 1 \in \mathbb{N}$ (since $b$ is a positive integer). Now, (3) yields \( \binom{a + b}{a} \) $\in \mathbb{Z}$ (since $a + b \in \mathbb{Z}$ and $a \in \mathbb{N}$) and \( \binom{a + b - 1}{a - 1} \) $\in \mathbb{Z}$ (since $a + b - 1 \in \mathbb{Z}$ and $a - 1 \in \mathbb{N}$) and \( \binom{a + b - 1}{b - 1} \) $\in \mathbb{Z}$ (since $a + b - 1 \in \mathbb{Z}$ and $b - 1 \in \mathbb{N}$).

Define $h \in \mathbb{Q}$ by $h = \frac{1}{a+b} \binom{a + b}{a}$. (This is well-defined, since $a + b$ is nonzero and \( \binom{a + b}{a} \) belongs to $\mathbb{Z}$.)

From $h = \frac{1}{a+b} \binom{a + b}{a}$, we obtain

$$ah = a \cdot \frac{1}{a+b} \binom{a + b}{a} = \frac{a}{a+b} \binom{a + b}{a} = \binom{a + b - 1}{a - 1}$$

(by part (a) of this exercise) $\in \mathbb{Z}$.

From $h = \frac{1}{a+b} \binom{a + b}{a}$, we also obtain

$$bh = b \cdot \frac{1}{a+b} \binom{a + b}{a} = \frac{b}{a+b} \binom{a + b}{a} = \binom{a + b - 1}{b - 1}$$

(by part (a) of this exercise) $\in \mathbb{Z}$.

Thus, part (b) of this exercise yields $h \in \mathbb{Z}$. In view of $h = \frac{1}{a+b} \binom{a + b}{a} = \frac{a + b}{a + b - 1}$, this rewrites as $\frac{\binom{a + b}{a}}{a + b} \in \mathbb{Z}$. In other words, $a + b \mid \binom{a + b}{a}$ (since $a + b$ is nonzero). This solves part (c) of the exercise.

(d) For example, setting $a = 2$ and $b = 2$ yields a counterexample, since $2 + 2 \nmid \binom{2 + 2}{2}$.

(In fact, $2 + 2 = 4 \nmid 6 = \binom{4}{2} = \binom{2 + 2}{2}$.)

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REFERENCES

See https://www-cs-faculty.stanford.edu/~knuth/gkp.html for errata.

The numbering of theorems and formulas in this link might shift when the project gets updated; for a “frozen” version whose numbering is guaranteed to match that in the citations above, see https://github.com/darijgr/detnotes/releases/tag/2019-01-10.