1) Throughout this exercise I will denote $0_{m \times n} = 0$.

a. **$S_1$: This is a subspace.** Clearly $0 \in S_1$ since $0 + 0 = 0 + 0$. Now assume that we have $a = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $b = \begin{pmatrix} w & x \\ y & z \end{pmatrix} \in S_1$. Then $a + b = \begin{pmatrix} a + w & b + x \\ c + y & d + z \end{pmatrix}$. Then we can see that $(a + w) + (d + z) = (a + d) + (w + z) = (b + c) + (x + y) = (b + x) + (c + y)$. So $a + b \in S_1$. It is also clear that $\lambda a \in S_1$.

b. **$S_2$: This is a subspace.** This space is extremely similar to $S_1$, except one could imagine that $b + c = 0$, although this is not actually a requirement. Hence a similar argument follows

**S3:** **This is not a subspace.** Consider $a = \begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix}$, $b = \begin{pmatrix} 3 & 7 \\ 0 & 0 \end{pmatrix}$ Then we clearly have $\det(a) = 0$ and $\det(b) = 0$. However, $a + b = \begin{pmatrix} 5 & 9 \\ 1 & 1 \end{pmatrix}$ and so $\det(a + b) = -4 \neq 0$.

**S4:** **This is not a subspace.** Consider the matrices $a = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $b = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. It is easy to check that $a^2 = b^2 = 0$. However, $a + b = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^2 = I_2$.

**S5:** **This is a subspace.** We are essentially solving the equation $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 3 \\ 1 \end{pmatrix} = 0$. This gives us the equations $3a + b = 0$ and $3c + d = 0$. Clearly we have $0 \in S_5$, and the arguments for sums and scales follow similarly from $S_1$.

**S6:** **This is a subspace.** We solve the equation:

$$\begin{pmatrix} 1 & 2 \\ c & d \end{pmatrix} \begin{pmatrix} a \\ 3 \end{pmatrix} = (a + 2c \\ b + 2d) \begin{pmatrix} 3 \\ 1 \end{pmatrix} = 3a + 6c + b + 2d = 0$$

Once again we have that $0 \in S_6$, and sums and scales follow quickly.

**b.** We will find spans for all 4 of the subspaces.

For $S_1$, we can let $b = r$, $c = s$ and $d = t$. Then we have $a = r + s - t$. So we have the following span:

$$S_1 = \langle E_{1,1} + E_{1,2}, E_{1,1} + E_{2,1}, -E_{1,1} + E_{2,2} \rangle$$

For $S_2$, we require that $a = -d$, while $b$ and $c$ can be anything. So we get the span:

$$\langle E_{1,1} - E_{2,2}, E_{1,2}, E_{2,1} \rangle$$

For $S_5$, as said in part (a) we require that $b = -3a$ and $d = -3c$. So we get the span:

$$\langle E_{1,2} - 3E_{1,1}, E_{2,2} - 3E_{2,1} \rangle$$

For $S_6$, we can have $b = -3a - 6c - 2d$, with $a, c, d$ being anything. Hence we get:

$$\langle E_{1,1} - 3E_{1,2}, E_{2,1} - 6E_{1,2}, E_{2,2} - 2E_{1,2} \rangle$$

2) **a.** Yes this does span all of $\mathbb{R}^3$. Note that $(1, 1, 0)^T - (0, 1, 1)^T + (1, 0, 1)^T = (2, 0, 0)^T$, $(1, 1, 0)^T + (0, 1, 1)^T - (1, 0, 1)^T = (0, 2, 0)^T$ and $(1, 0, 1)^T + (0, 1, 1)^T - (1, 1, 0)^T = (0, 0, 2)^T$. So for any
(a, b, c)^T \in \mathbb{R}^3$, we can write it as $(a/2)(2, 0, 0)^T + (b/2)(0, 2, 0)^T + (c/2)(0, 0, 2)^T$. So $(a, b, c)^T$ is in the span.

b. This does not span all of $\mathbb{R}^4$. Consider the vector $(1, 0, 1, 0)^T \in \mathbb{R}^4$. Then we require that:

$$\lambda_1 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \lambda_3 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \lambda_4 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

So we get the following system:

$$\begin{align*}
\lambda_1 + \lambda_4 &= 1 \\
\lambda_1 + \lambda_2 &= 0 \\
\lambda_2 + \lambda_3 &= 1 \\
\lambda_3 + \lambda_4 &= 0
\end{align*}$$

Hence we get that $\lambda_1 = -\lambda_2$ and $\lambda_3 = -\lambda_4$. Thus we get the equations $\lambda_4 - \lambda_2 = 1$ and $\lambda_2 - \lambda_4 = 1$. Clearly this is not solvable system, so there are no such $\lambda_i$ satisfy the matrix equation, so $(1, 0, 1, 0)^T$ is not in the span, so the given matrices do not span $\mathbb{R}^4$.

3) The blanks are filled with $(u_1 + u_2 + \cdots + u_j) \in U$ (from the induction hypothesis) and $u_{j+1} \in U$. So this is indeed the sum of two vectors of $U$.

4) (Step 1): The vector $\lambda x = (\lambda x_1, \lambda x_2, \ldots, \lambda x_n)^T$ satisfies $\lambda x_1 + \lambda x_2 + \cdots + \lambda x_n = \lambda(x_1 + x_2 + \cdots + x_n) = \lambda(0) = 0$.

(Step 8): We have $X$ as a subspace of $\mathbb{R}^n$ since it is the span of vectors that are in $\mathbb{R}^n$, and any span of vectors in a space create a subspace of that space.

Applying Proposition 0.1 (c) to $U = X$, $k = n - 1$ and $u_i = e_i - e_n$.

5) a. We are looking to solve for $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ such that $(2, 3, 1, -2, -4)^T = \lambda_1(1, 0, 0, 0, -1)^T + \lambda_2(0, 1, 0, 0, -1)^T + \lambda_3(0, 0, 1, 0, -1)^T + (0, 0, 0, 1, -1)^T$. We can read off almost immediately that $\lambda_1 = 2$, $\lambda_2 = 3$, $\lambda_3 = 1$ and $\lambda_4 = -2$. Then we see that $-(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4) = -(2 + 3 + 1 - 2) = -4$ as expected. Hence:

$$(2, 3, 1, -2, -4)^T = 2(e_1 - e_5) + 3(e_2 - e_5) + 1(e_3 - e_5) - 2(e_4 - e_5)$$

b. We are solving $(2, 3, 1, -2, -4)^T = \lambda_1(1, -1, 0, 0, 0)^T + \lambda_2(0, 1, -1, 0, 0)^T + \lambda_3(0, 0, 1, -1, 0)^T + \lambda_4(0, 0, 0, 1, -1)^T$ for $\lambda_1, \lambda_2, \lambda_3, \lambda_4$. We get the following system:

$$\begin{align*}
\lambda_1 + 0 &= 2 \\
\lambda_2 - \lambda_1 &= 3 \\
\lambda_3 - \lambda_2 &= 1 \\
\lambda_4 - \lambda_3 &= -2 \\
0 - \lambda_4 &= -4
\end{align*}$$

Solving this through substitution is rather simple. We get $\lambda_1 = 2, \lambda_2 = 5, \lambda_3 = 6, \lambda_4 = 4$. Hence:

$$(2, 3, 1, -2, -4)^T = 2(e_1 - e_2) + 5(e_2 - e_3) + 6(e_3 - e_4) + 4(e_4 - e_5)$$

6) a. We wish to determine if there are $\lambda_1, \lambda_2, \lambda_3$ where not all three $\lambda_i = 0$, such that $\lambda_1(1, 1, 0)^T + \lambda_2(0, 1, 1)^T + \lambda_3(1, 0, 1)^T = 0$. In other words, we want to solve the system:

$$\begin{align*}
\lambda_1 + 0 + \lambda_3 &= 0 \\
\lambda_1 + \lambda_2 + 0 &= 0 \\
0 + \lambda_2 + \lambda_3 &= 0
\end{align*}$$

From a non-rigorous standpoint, we have three equations and three variables so we expect there to be only one solution. Solving through, we get $\lambda_2 = -\lambda_3$, so we have $\lambda_1 - \lambda_3 = 0$ and $\lambda_1 + \lambda_3 = 0$. Hence we must have $\lambda_i = 0$ for $i = 1, 2, 3$. Thus these vectors linearly independent.
b. These are not linearly independent. We can take:

\[
\begin{pmatrix}
1 \\
1 \\
0 \\
0
\end{pmatrix} - \begin{pmatrix}
0 \\
1 \\
1 \\
0
\end{pmatrix} + \begin{pmatrix}
0 \\
0 \\
1 \\
1
\end{pmatrix} - \begin{pmatrix}
1 \\
0 \\
0 \\
0
\end{pmatrix} = \begin{pmatrix}
0 \\
0 \\
0 \\
0
\end{pmatrix}
\]

Hence it is possible to obtain 0 without setting all coefficients of the vectors to 0, meaning they are linearly dependent.

7) 1: We can divide the inequality by \( \lambda_i \) because we chose \( \lambda_i \neq 0 \).
2: \( \{v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_k\} \)
3: \( \{v_1, \ldots, v_k\} \)

8) a. The only such \( i \) is \( i = 4 \), as \( v_1, v_2 \) and \( v_3 \) are linearly independent. We can write \( v_4 = v_1 + 2v_2 + 3v_3 \).
   b. We can take \( i = 3, 4 \), since \( v_3 = 2v_1 - v_2 \) and \( v_4 = -v_1 + v_2 + 2v_3 \).
   c. We can only take \( i = 1 \), because \( v_2 = 0 \) is a linear combination of (i).