Let me first recall a definition.

**Definition 0.1.** Let $V$ and $W$ be two vector spaces. Let $v = (v_1, v_2, \ldots, v_m)$ be a basis of $V$. Let $w = (w_1, w_2, \ldots, w_n)$ be a basis of $W$. Let $L : V \rightarrow W$ be a linear map.

The *matrix representing* $L$ with respect to $v$ and $w$ is the $n \times m$-matrix $M_{v,w,L}$ defined as follows: For every $j \in \{1, 2, \ldots, m\}$, expand the vector $L(v_j)$ with respect to the basis $w$, say, as follows:

$$L(v_j) = \alpha_{1,j}w_1 + \alpha_{2,j}w_2 + \cdots + \alpha_{n,j}w_n. \quad (1)$$

Then, $M_{v,w,L}$ is the $n \times m$-matrix whose $(i, j)$-th entry is $\alpha_{i,j}$.

For example, if $n = 3$ and $m = 2$, then

$$M_{v,w,L} = \begin{pmatrix} \alpha_{1,1} & \alpha_{1,2} \\ \alpha_{2,1} & \alpha_{2,2} \\ \alpha_{3,1} & \alpha_{3,2} \end{pmatrix},$$

where

$$L(v_1) = \alpha_{1,1}w_1 + \alpha_{2,1}w_2 + \alpha_{3,1}w_3;$$

$$L(v_2) = \alpha_{1,2}w_1 + \alpha_{2,2}w_2 + \alpha_{3,2}w_3.$$

The purpose of this matrix $M_{v,w,L}$ is to allow easily expanding $L(v)$ in the basis $(w_1, w_2, \ldots, w_n)$ of $W$ if $v$ is a vector in $V$ whose expansion in the basis $(v_1, v_2, \ldots, v_m)$ of $V$ is known. For instance, if $v$ is one of the basis vectors $v_j$, then the expansion of $L(v_j)$ can be simply read off from the $j$-th column of $M_{v,w,L}$; otherwise, it is an appropriate linear combination:

$$L(\lambda_1v_1 + \lambda_2v_2 + \cdots + \lambda_mv_m) = \lambda_1L(v_1) + \lambda_2L(v_2) + \cdots + \lambda_mL(v_m)$$

(where the $L(v_j)$ can be computed by (1)).

You can abbreviate $M_{v,w,L}$ as $M_L$, but it’s your job to ensure that you know what $v$ and $w$ are (and they aren’t changing midway through your work).

**Example 0.2.** Let $A$ be the $2 \times 2$-matrix $\begin{pmatrix} 5 & 7 \\ -2 & 9 \end{pmatrix}$. Let $L : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be the linear map $L_A$. (Recall that this is the map $\mathbb{R}^2 \rightarrow \mathbb{R}^2$ that sends every vector $v \in \mathbb{R}^2$ to $Av$.)
Consider the following basis \( v = (v_1, v_2) \) of the vector space \( \mathbb{R}^2 \):

\[
v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}.
\]

Consider the following basis \( w = (w_1, w_2) \) of the vector space \( \mathbb{R}^2 \):

\[
w_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad w_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.
\]

What is the matrix \( M_{v,w,L} \) representing \( L \) with respect to these two bases \( v \) and \( w \)?

First, let me notice that it is not \( A \) (or at least it doesn’t have to be \( A \) a priori), because our two bases \( v \) and \( w \) are not the standard basis of \( \mathbb{R}^2 \). Only if we pick both \( v \) and \( w \) to be the standard bases of the respective spaces we can guarantee that \( M_{v,w,L} \) will be \( A \).

Without having this shortcut, we must resort to the definition of \( M_{v,w,L} \). It tells us to expand \( L(v_1) \) and \( L(v_2) \) in the basis \( w \) of \( \mathbb{R}^2 \), and to place the resulting coefficients in a \( 2 \times 2 \)-matrix. Let’s do this. We begin with \( L(v_1) \):

\[
L(v_1) = L_A(v_1) = Av_1 = \begin{pmatrix} 5 & 7 \\ -2 & 9 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -2 \\ -11 \end{pmatrix}.
\]

How do we expand this in the basis \( w \)? This is a typical exercise in Gaussian elimination (we just need to solve the equation \( L(v_1) = \lambda_1 w_1 + \lambda_2 w_2 \) in the two unknowns \( \lambda_1 \) and \( \lambda_2 \), and the result is

\[
L(v_1) = \frac{7}{3} w_1 + \frac{-20}{3} w_2.
\]

Similarly, we take care of \( L(v_2) \), obtaining

\[
L(v_2) = 13w_1 + 5w_2.
\]

Thus, the required matrix is

\[
M_{v,w,L} = \begin{pmatrix} 7 & 13 \\ \frac{3}{3} & \frac{-20}{3} \\ \frac{-20}{3} & 5 \end{pmatrix}.
\]

**Exercise 1.** Let \( A \) be the \( 3 \times 2 \)-matrix \( \begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix} \). Let \( L : \mathbb{R}^3 \to \mathbb{R}^2 \) be the linear map \( L_A \). (Recall that this is the map \( \mathbb{R}^3 \to \mathbb{R}^2 \) that sends every vector \( \mathbf{v} \in \mathbb{R}^3 \) to \( Av \).)
Consider the following basis \( v = (v_1, v_2, v_3) \) of the vector space \( \mathbb{R}^3 \):

\[
\begin{align*}
v_1 &= \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \\
v_2 &= \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \\
v_3 &= \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.
\end{align*}
\]

Consider the following basis \( w = (w_1, w_2) \) of the vector space \( \mathbb{R}^2 \):

\[
\begin{align*}
w_1 &= \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \\
w_2 &= \begin{pmatrix} 1 \\ 0 \end{pmatrix}.
\end{align*}
\]

(a) Find the matrix \( M_{v,w,L} \) representing \( L \) with respect to these two bases \( v \) and \( w \). \([15 \text{ points}]\)

(b) Let \( v' \) be the basis \( (v_3, v_2, v_1) \) of \( \mathbb{R}^3 \). Let \( w' \) be the basis \( (w_2, w_1) \) of \( \mathbb{R}^2 \). Find the matrix \( M_{v',w',L} \). \([5 \text{ points}]\)

For every \( n \in \mathbb{N} \), we let \( P_n \) denote the vector space of all polynomial functions (with real coefficients) of degree \( \leq n \) in one variable \( x \). This vector space has dimension \( n + 1 \), and its simplest basis is \( (1, x, x^2, \ldots, x^n) \). We call this basis the **monomial basis** of \( P_n \).

If \( f \) is a polynomial in one variable \( x \), then I shall use the notation \( f[y] \) for “substitute \( y \) for \( x \) into \( f \)”. (For example, if \( f = x^3 + 7x + 2 \), then \( f[5] = 5^3 + 7 \cdot 5 + 2 = 162 \).) This would normally be denoted by \( f(y) \), but this is somewhat ambiguous, since the notation \( x(x + 1) \) could then stand for two different things (namely, “substitute \( x + 1 \) into the polynomial function \( x^n \)” or “multiply \( x \) by \( x + 1 \)”), whereas the notation \( f[y] \) removes this ambiguity.

**Example 0.3.**

(a) Define a map \( S_a : P_2 \to \mathbb{R} \) by \( S_a(f) = f[2] + f[3] \). Then, \( S_a \) is linear, because:

1. If \( f \) and \( g \) are two elements of \( P_2 \), then

\[
S_a(f + g) = (f + g)[2] + (f + g)[3] = (f[2] + g[2]) + (f[3] + g[3])
\]

\[
= S_a(f) + S_a(g).
\]

2. If \( f \in P_2 \) and \( \lambda \in \mathbb{R} \), then

\[
S_a(\lambda f) = (\lambda f)[2] = \lambda f[2] = \lambda S_a(f).
\]

Let \( v \) be the monomial basis \( (1, x, x^2) \) of \( P_2 \), and let \( w \) be the one-element basis \( (1) \) of \( \mathbb{R} \). What is the matrix \( M_{v,w,S_a} \)?
Again, follow the definition of $M_{v,w,S_a}$. It tells us to expand $S_a(1)$, $S_a(x)$ and $S_a(x^2)$ in the basis $w$ of $\mathbb{R}$, and to place the resulting coefficients in a $1 \times 3$-matrix. Expanding things in the basis $w$ is particularly simple, since $w$ is a one-element list; specifically, we obtain the expansions

\[
S_a(1) = 1 \begin{bmatrix} 2 \\ =1 \end{bmatrix} + 1 \begin{bmatrix} 3 \\ =1 \end{bmatrix} = 1 + 1 = 2 = 2 \cdot 1;
\]
\[
S_a(x) = x \begin{bmatrix} 2 \\ =2 \end{bmatrix} + x \begin{bmatrix} 3 \\ =3 \end{bmatrix} = 2 + 3 = 5 = 5 \cdot 1;
\]
\[
S_a(x^2) = x^2 \begin{bmatrix} 2 \\ =2^2 \end{bmatrix} + x^2 \begin{bmatrix} 3 \\ =3^2 \end{bmatrix} = 2^2 + 3^2 = 13 = 13 \cdot 1.
\]

Thus, the required matrix is

\[
M_{v,w,S_a} = \begin{pmatrix} 2 & 5 & 13 \end{pmatrix}.
\]

(b) Define a map $S_b : P_2 \to P_4$ by $S_b(f) = f[x^2]$. (Notice that we chose $P_4$ as the target space, because substituting $x^2$ for $x$ will double the degree of a polynomial.) The map $S_b$ is linear (for reasons that are similar to the ones that convinced us that $S_a$ is linear).

Let $v$ be the monomial basis $(1, x, x^2)$ of $P_2$, and let $w$ be the monomial basis $(1, x, x^2, x^3, x^4)$ of $P_4$. What is the matrix $M_{v,w,S_b}$?

Again, follow the definition of $M_{v,w,S_b}$. It tells us to expand $S_b(1)$, $S_b(x)$ and $S_b(x^2)$ in the basis $w$ of $P_4$, and to place the resulting coefficients in a $5 \times 3$-matrix. The expansions are as follows:

\[
S_b(1) = 1 \begin{bmatrix} x^2 \\ =1 \end{bmatrix} = 1 = 1 \cdot 1 + 0 \cdot x + 0 \cdot x^2 + 0 \cdot x^3 + 0 \cdot x^4;
\]
\[
S_b(x) = x \begin{bmatrix} x^2 \\ =1 \end{bmatrix} = x^2 = 0 \cdot 1 + 0 \cdot x + 1 \cdot x^2 + 0 \cdot x^3 + 0 \cdot x^4;
\]
\[
S_b(x^2) = x^2 \begin{bmatrix} x^2 \\ =1 \end{bmatrix} = (x^2)^2 = x^4 = 0 \cdot 1 + 0 \cdot x + 0 \cdot x^2 + 0 \cdot x^3 + 1 \cdot x^4.
\]

Thus, the required matrix is

\[
M_{v,w,S_b} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.
\]

Exercise 2. Which of the following maps are linear? For every one that is, represent it as a matrix with respect to the monomial bases of its domain and its
target.  [6 points for each part, split into 2+4 if the map is linear]

(a) The map $T_a : P_2 \to P_2$ given by $T_a(f) = f[x+1]$. (Thus, $T_a$ is the map that substitutes $x+1$ for $x$ into $f$. Thus, $T_a(x^n) = (x+1)^n$ for every $n \in \{0, 1, 2\}$.)

(b) The map $T_b : P_2 \to P_3$ given by $T_b(f) = xf[x]$. (Notice that $f[x]$ is the same as $f$, because substituting $x$ for $x$ changes nothing. I am just writing $f[x]$ to stress that $f$ is a function of $x$.)

(c) The map $T_c : P_2 \to P_4$ given by $T_c(f) = f[1]f[x]$.

(d) The map $T_d : P_2 \to P_4$ given by $T_d(f) = f[x^2 + 1]$.

(e) The map $T_e : P_2 \to P_2$ given by $T_e(f) = x^2f\left[\frac{1}{x}\right]$.

(g) The map $T_g : P_3 \to P_3$ given by $T_g(f) = xf'[x]$.

[There is no part (f) because I want to avoid calling a map “$T_f$” while the letter $f$ stands for a polynomial.]

[Note: Proofs are not required.]

See the beginning of §3.21 of [my notes](link), the [Wikipedia](link) or various other sources, for examples of injective, surjective and bijective maps.

**Exercise 3.**

(a) Which of the six maps in Exercise 2 are injective?  [2 points per map]

(b) Which of them are surjective?  [2 points per map]

[Note: Proofs are not required.]