Math 4242 Fall 2016 (Darij Grinberg): midterm 2 practice problems

Exercise 1. Consider the vector space $\mathbb{R}^3$.
(a) The list $a = \left( (1,2,-1)^T, (1,1,0)^T, (0,1,-1)^T, (1,1,1)^T \right)$ spans $\mathbb{R}^3$. Shrink this list to a basis of $\mathbb{R}^3$ by removing some redundant elements.
(b) The list $b = \left( (-1,0,1)^T, (2,3,4)^T \right)$ is linearly independent. Extend this list to a basis of $\mathbb{R}^3$ by appending to it some elements from the list $a$.

Exercise 2. (a) Find bases of the four subspaces of the $3 \times 4$-matrix $A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 \\ 1 & 2 & 3 & 3 \end{pmatrix}$.
(b) [Too tricky for a midterm, but worth thinking about!] More generally: Let $n \in \mathbb{N}$ and $m \in \mathbb{N}$. Let $A_{n \times m}$ be the $n \times m$-matrix $(\min \{i,j\})_{1 \leq i \leq n, \ 1 \leq j \leq m}$. (This is the $n \times m$-matrix whose $(i,j)$-th entry is $\min \{i,j\}$.) For example, $A_{3,4}$ is the matrix $A$ from part (a) of this exercise.
Find bases of the four subspaces of $A_{n \times m}$.

If $A$ is an $n \times k$-matrix whose columns are linearly independent, then a QR decomposition of $A$ means a way to write $A$ in the form $A = QR$, where:
- $Q$ is an $n \times k$-matrix with orthonormal columns (this is equivalent to saying that $Q$ is an $n \times k$-matrix satisfying $Q^TQ = I_k$);
- $R$ is an upper-triangular $k \times k$-matrix with nonzero diagonal entries.

For example, a QR decomposition of $\begin{pmatrix} 2 & 17 \\ 4 & 13 \\ 8 & 5 \end{pmatrix}$ is
\[
\begin{pmatrix}
2 & 17 \\
4 & 13 \\
8 & 5
\end{pmatrix} = \begin{pmatrix}
1 & 2 \\
\frac{2}{\sqrt{21}} & \frac{1}{\sqrt{21}} \\
\frac{4}{\sqrt{21}} & -\frac{1}{\sqrt{6}}
\end{pmatrix}
\begin{pmatrix}
2\sqrt{21} & 3\sqrt{21} \\
0 & 7\sqrt{6}
\end{pmatrix}.
\]

this is the $Q$

this is the $R$

Exercise 3. (a) Find a QR decomposition of the matrix $\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$.
(b) Find a QR decomposition of the matrix $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$.
(c) Find a QR decomposition of the matrix \( \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \).

[Hint: Two of the three parts are easy and can be done with no computations whatsoever!]

Exercise 4. (a) Apply the Gram-Schmidt process to the two vectors
\[ w_1 = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \quad w_2 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \]
in \( \mathbb{R}^3 \).

(b) Let \( U \) be the subspace of \( \mathbb{R}^3 \) spanned by \( w_1, w_2 \). Find the projection \( u \) of the vector \( b = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \) on the subspace \( U \).

Exercise 5. Find the least-squares solution to the equation \( Ax = b \), where
\[
A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}.
\]

Exercise 6. Let \( p \in \mathbb{N} \). Find the least-squares solution \( x \in \mathbb{R}^2 \) to the equation
\[
Ax = b, \quad \text{where} \quad A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ \vdots & \vdots \\ 1 & 2 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ -1 \\ 0 \end{pmatrix}.
\]
(The matrix \( A \) has \( p + 2 \) rows and 2 columns, and the column vector \( b \) has size \( p + 2 \). All entries of \( A \) are 1’s except for the last two entries of the second column. All entries of \( b \) are 1 except for the last two entries.)

(For example, if \( p = 3 \), then
\[
A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 2 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \end{pmatrix}, \quad \text{and the least-squares solution is}
\[
\begin{pmatrix} \frac{9}{10} \\ -\frac{1}{2} \end{pmatrix}.
\]
[Feel free to check your result visually: This exercise is a data-fitting problem,
where you are trying to fit a line $y = \alpha t + \beta$ through the $p + 2$ points

$$(1,1), (1,1), \ldots, (1,1), (2,-1), (0,0).$$

Thus, the least-squares solution $x = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ should lead to a line $y = \alpha + \beta t$ that comes relatively close to all these points, but gets pulled closer and closer to $(1,1)$ when $p$ grows (because with growing $p$, the point $(1,1)$ gets repeated more often and thus “pulls more weight”).]