Prop. 3.12. Let \( n \in \mathbb{N} \) and \( k \in \mathbb{N} \).

(a) We have \( \{0\} = \{n=0\} \).

(b) We have \( \{n\} = \{k=0\} \).

(c) \( \{n\} = 0 \) if \( k > n \).

(d) \( \{n\} = \{k-1\} + k \{n\} \) if \( n > 0 \) & \( k > 0 \).

(e) \( \{n\} = \sum_{j=0}^{n-1} \binom{n}{j} \{k-j\} / k \).

(f) \( \{k\} = \frac{1}{k!} \sum_{c=0}^{k} (-1)^{k-c} \binom{k}{c} n^c \).

Pf. Follows from Prop. 2.10, Prop. 2.14, Prop. 2.12, Thm. 2.14.

Remark: Let \( p \) be a prime number.

We know: \( p | \{k\} \) \( \forall k \in \{1, 2, ..., p-1\} \). (This is Thm. 1.24.)

But also \( p | \{k\} \) \( \forall k \in \{2, 3, ..., p-1\} \). Why?
Let us reprove $p|k$.

Let $P_k([n])$ be the set of all $k$-element subsets of $[n]$. We define the shift of some $X \in P_k([n])$ to be the subset obtained from $X$ by replacing $1, 2, \ldots, n-1, n$ by $2, 3, \ldots, n, 1$.

(For example, if $n=6$, then the shift of $\{2, 3, 5\}$ is $\{3, 4, 6\}$, and $\{2, 3, 6\}$ is $\{3, 4, 1\}$.)

Now, let $n=p$ be prime and $k \in \{1, \ldots, p-1\}$. Define an equivalence relation $\sim$ on $P_k([n])$ by letting $X \sim Y$ if and only if $Y$ can be obtained from $X$ by shifts. E.g. for $n=6$, we have

$\{2, 3, 5\} \sim \{3, 4, 6\} \sim \{4, 5, 1\} \sim \{5, 6, 2\}$

$\{6, 1, 3\} \sim \{1, 2, 4\} \sim \{2, 3, 5\}$. 
Any \( \sim \)-equivalence class has \( \leq n \) elements, since shifting a subset \( n \) times brings it back home.

In general, it can have \( < n \) elements:

E.g., if \( n = 6 \) & \( k = 3 \), then \( \{3, 6\} \sim \{4, 1\} \sim \{5, 2\} \sim \{6, 3\} \)

so the class only has 3 elements.

But if \( n = p \) prime & \( k \in \{1, 2, \ldots, p-1\} \), it will always have exactly \( n = p \) elements (easy algebra: need to check that the shift of \( S \in \mathcal{P}_k([n]) \) is not \( S \) itself).

Thus, the relation \( \sim \) subdivides the set \( \mathcal{P}_k([n]) \) into \( \sim \)-equivalence classes, and each of these classes has size \( p \).

Thus, \( \left| \mathcal{P}_k([n]) \right| = (\# \text{ of } \sim \text{-equiv. classes}) \cdot p \).

Hence, \( p \mid \left| \mathcal{P}_k([n]) \right| = \binom{n}{k} = \binom{p}{k} \).
The proof of $p \mid \binom{r}{k}$ (for $k \in \{2, 3, \ldots, p-1\}$) is similar.

**Remark.** Let $n \in \mathbb{N}$. The Bell number $B(n)$ is the # of all set partitions of $[n]$. Thus, $B(n) = \binom{n}{0} + \binom{n}{1} + \ldots + \binom{n}{n}$.

It satisfies $B(n+1) = \sum_{i=0}^{n} \binom{n}{i} B(i)$. (Exercise.)

See Spring 2018 Math 4707 for more.

3.6. $U \rightarrow U$ and integer partitions

Idea: a $U \rightarrow U$ placement looks like this:

```
1000
1 1 1 1
1 1 1 1
```

but the boxes, too, are interchangeable.

Thus, we can always order the boxes by decreasing number of balls:

```
1000
1 1 1 1
```

You can encode this U→U placement by a sequence of numbers, telling how many balls each box gets:

\((3, 2, 2, 1, 0, 0, 0, 0)\).

The decreasing order makes the sequence unique.

**Def.** A partition of an integer \(n\) is a weakly decreasing list \((a_1, a_2, \ldots, a_k)\) of positive integers whose sum is \(n\) (that is, \(a_1 \geq a_2 \geq \ldots \geq a_k > 0\) and \(a_1 + a_2 + \ldots + a_k = n\)).

The integers \(a_1, a_2, \ldots, a_k\) are called the parts of the partition.

**Example:** The partitions of 5 are

\((5), (4,1), (3,2), (3,1,1), (2,2,1), (2,1,1,1), (1,1,1,1,1)\).

**Rmk:** A partition is the same as a weakly decreasing composition.
Def. Let \( n \in \mathbb{Z} \) and \( k \in \mathbb{N} \). Then, \( p_k(n) \) means the number of partitions of \( n \) into \( k \) parts (= having \( k \) parts).

Examples: \( p_0(5) = 0, \quad p_1(5) = 1, \quad p_2(5) = 2, \quad p_3(5) = 2, \quad p_4(5) = 1, \quad p_5(5) = 1, \quad p_k(5) = 0 \quad \forall k > 5 \).

Prop. 3.13. (a) \( p_k(n) = 0 \quad \forall n < 0 \).
(b) \( p_k(n) = 0 \) if \( k > n \).
(c) \( p_0(n) = \lfloor n = 0 \rfloor \).
(d) \( p_1(n) = \lfloor n > 0 \rfloor \).
(e) \( p_k(n) = p_k(n-k) + p_{k-1}(n-1) \quad \forall n \in \mathbb{Z} \) & \( k \geq 1 \).
(f) \( p_2(n) = \lfloor n/2 \rfloor \quad \forall n \geq 0 \).

Proof. (a) A sum of positive integers is never negative.
(b) A partition into \( k \) parts has sum \( \geq \underbrace{1+1+\ldots+1}_k = k \).
(c) The only partition into 0 parts is ()

(d) The only partition of n into 1 part is (n), which only exists if n > 0.

(e) Classify the partitions of n into k parts into 2 types:
   - Type 1: partitions that have 1 as a part
   - Type 2: partitions that don't

There is a bijection

\[ \{ \text{Type-1 partitions of } n \text{ into } k \text{ parts} \} \]
\[ \rightarrow \{ \text{partitions of } n-1 \text{ into } k-1 \text{ parts} \} \]

\[ (\lambda_1, \lambda_2, \ldots, \lambda_{k-1}) \rightarrow (\lambda_1, \lambda_2, \ldots, \lambda_{k-1}) \]

(because any Type-1 partition must end with a 1).

Thus, \( \# \text{ of Type-1 partitions} = \binom{k-1}{n-1} \).

Also, there is a bijection

\[ \{ \text{Type-2 partitions of } n \text{ into } k \text{ parts} \} \]
\[ \rightarrow \{ \text{partitions of } n-k \text{ into } k \text{ parts} \} \]

\[ (\lambda_1, \lambda_2, \ldots, \lambda_k) \rightarrow (\lambda_1-1, \lambda_2-1, \ldots, \lambda_{k-1}) \]
Thus, \((\#\text{of Type}-2\text{ partitions}) = p_k(n-k)\).

Adding these equalities together, we get the claim of (e).

(f) The partitions of \(n\) into 2 parts are
\[(n-1, 1), \ (n-2, 2), \ (n-3, 3), \ldots, \ \left(\lceil n/2 \rceil, \lfloor n/2 \rfloor\right), \]

the ceiling function

\[\Box\]

Prop. 3.14. \((\#\text{ of surjective } U\rightarrow U \text{ placements } A \rightarrow X) = p_{|X|}(|A|)\),
Proof idea. Encode 2 surjective \(U\rightarrow U\) placement as a partition of \(|A|\) into \(|X|\) parts: namely, the partition (\# of balls in the box with the largest \# of balls, second-largest \(--\), third-largest \(--\),

\[
\begin{array}{cc}
\# & \# \\
\hline
\# & \# \\
\hline
\end{array}
\]

This is a bijection. \[\Box\]
Prop. 3.15. \((\# \text{ of } U \rightarrow U \text{ placements}) = p_0(1A1) + p_2(1A1) + \ldots + p_{1x1}(1A1)\) 
\[= p_{1x1}(1A1 + 1x1),\] 

Ex: \(|A1| = 3, \ |X1| = 4\) : 

\[\begin{array}{cccc}
\cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots \\
\end{array}\]

\[p_4(3+4) = p_4(3) + p_3(6)\] 
\[= p_3(6) = p_3(3) + p_2(5)\] 
\[= 1 + p_2(5) = 1 + \frac{5}{2} = \frac{7}{2} = 3\]

Proof. Exercise

Prop. 3.16. \((\# \text{ of injective } U \rightarrow U \text{ placements}) = [1A1 \leq |X1|]\)

Proof. Exercise

\[\square\]
Exercise: Given \( n \) persons (\( n > 0 \)) and \( k \) tasks (\( k > 0 \)).

(a) What is the number of ways to assign a task to each person such that each task has at least 1 person working on it?

(b) What if we additionally want to choose a leader for each task (among the people assigned to it)?

(c) What if, instead, we want to choose a vertical hierarchy (between all people working on the task) for each task?

Example: 8 people (1, 2, 3, ..., 8) and 3 tasks (A, B, C).

(a): \( 1 \ 2 \ 5 \)  \( 3 \)  \( 4 \ 6 \ 7 \ 8 \)

(b): \( 1 \ 2 \ 5 \)  \( 3 \)  \( 4 \ 6 \ 7 \ 8 \)
(c): \[ \begin{array}{c}
1 \\
2 \\
5
\end{array} \]
\[ A \]

[Diagram: Circle with numbers 1, 2, 5, and a circle with numbers 3, 4, 8, 7, 16, 6, labeled B and C.]

Answer:
(a) sur \((n, k)\).

(b) \[ \binom{n}{k} \cdot k^{n-k} \]
(Pick the \(n\) leaders first, then assign the rest of the people arbitrarily.)

(c) \[ n! \cdot (n-1) \]
(First, order the entire \(n\) people, then split the ordering into \(k\) nonempty chunks)

(See Fall 2017 Math 4990 for details.)
Exercise: Let $X$ be an $n$-element set.

(a) What is the # of triples $(A, B, C)$ of subsets of $X$ such that $B \cap C = \emptyset$ and $C \cap A = \emptyset$ and $A \cap B = \emptyset$?

(b) What is the # of triples $(A, B, C)$ of subsets of $X$ such that $A \cap B \cap C = \emptyset$?

(See also fall 2017 Math 4550 hw #3 Exercise 2.)

Answers:

(a) $4^n$.
   (For each element $x \in X$, we choose to put $x$ in $A$ or in $B$ or in $C$ or in none of $A, B, C$; these are $4$ options.)

(b) $7^n$.
   (Now there are $7$ options for each $x \in X$.)
Exercise: Let $n \in \mathbb{N}$. A subset $S$ of $[2n]$ is called shadowed if all odd $i$s we have $i \in S$.

How many shadowed subsets does $[2n]$ have?

Answer: $3^n$.

(For each $k \in [n]$, we choose between having
- $2k-1 \in S$, $2k \notin S$;
- $2k-1 \notin S$, $2k \in S$;)

Exercise: Let $n \in \mathbb{N}$. How many compositions of $n$ have the property that all entries of the composition are in $\{1, 2\}$?

$n = 5 \Rightarrow (1, 1, 1, 1, 1), (1, 1, 1, 2), (1, 1, 2, 1), (1, 2, 1, 1), (2, 1, 1, 2), (2, 2, 1), (2, 1, 2), (1, 2, 2)$. 

Answer: the Fibonacci number $f_{n+1}$. 
(Proof 1: induction on $n$.
Proof 2: bijection to lowerer subsets.
Proof 3:

\[ 8 = 1 + 2 + 1 + 2 + 2 + 1 \]

Exercise: Let $n \in \mathbb{N}$ and $d > 0$.

An $n$-tuple $(x_1, x_2, \ldots, x_n) \in [d]^n$ is called 1-even if the number of i's satisfying $x_i = 1$ is even.

(e.g. $(5, 4, 1, 2, 1)$ and $(5, 4, 2)$ are 1-even, but $(4, 2, 4, 2)$ is not.)

What is the number of 1-even $n$-tuples?

(This is Spring 2013 Math 4707 homework set #3 exercise 5.)
Answer: \[
\sum_{k=0}^{\lfloor n/2 \rfloor} {n \choose 2k} (d-1)^{n-2k}
\]
(because first choose how many 1's the \( n \)-tuple will have;
then choose where to put them;
then choose each of the remaining \( n-2k \)
entries from \( d-1 \) choices)

\[
= \sum_{k \text{ even}} {n \choose k} (d-1)^{n-k} = \frac{1}{2} (d^n + (d-2)^n).
\]

More generally,
\[
\sum_{k \text{ even}} {n \choose k} x^k y^{n-k} = \frac{1}{2} ((x+y)^n + (-x+y)^n)
\]
(by adding \[
\sum_{k \in \mathbb{Z}} {n \choose k} x^k y^{n-k} = (x+y)^n
\]
with \[
\sum_{k \in \mathbb{Z}} {n \choose k} (-x)^k y^{n-k} = (-x+y)^n
\] )
Next hw will have an exercise on counting all even \( n \)-tuples: those in which each number occurs an even # of times.

(see also Stanley's "Algebraic Combinatorics").