Remark: Let $p$ be a prime number.

Then, $p | \left[ \frac{n}{k} \right]$ for all $k \in \{2, 3, \ldots, p-1\}$.

(Cf. remark after Prop. 3.12.)

Proof outline. Similar to the Remark after Prop. 3.12:

For any $\sigma \in S_n$, we define the shift of $\sigma$ by replacing $1, 2, \ldots, n-1, n$ by $2, 3, \ldots, n, 1$ in the DCD of $\sigma$.

Thus, if $n = 7$ and $\sigma = \text{cyc}_{3, 4, 7} \circ \text{cyc}_{2, 6} \circ \text{cyc}_{1, 5} \circ \text{cyc}_5$,

then $\text{shift}(\sigma) = \text{cyc}_{4, 5, 1} \circ \text{cyc}_{3, 7} \circ \text{cyc}_2 \circ \text{cyc}_6$.

Again, define equivalence relation $\equiv_{\text{shift}}$.

Again, it's clear if $\sigma$ has $k$ cycles, then so does $\text{shift}(\sigma)$.

The $\text{shift}$-equivalence classes of permutations $\sigma \in S_p$ with $k$ cycles have size $p$. (Again, this follows from algebra; once you know that $\sigma \equiv \text{shift}(\sigma)$, all $\sigma \in S_p$)
with $k$ cycles. The latter fact can be proven by contradiction: If we had $\sigma = \text{shift}(\sigma)$, then $\forall i \in [n]$, $\sigma(i) = i$, the number $i^+ := \begin{cases} i + 1, & \text{if } i < n \\ 1, & \text{if } i = n \end{cases}$ would lie in a $\sigma$-cycle of the same length as the one $i$ lies in. This implies all the numbers $1, 2, \ldots, n$ lie in cycles of the same length $\Rightarrow$ all cycles have the same length $\Rightarrow$ all cycles have length 1 or $p$ (since $n = p$ is prime) $\Rightarrow$ there are $p$ or 1 cycles; but this contradicts $k = \{2, 3, \ldots, p^2\}$ $\Rightarrow$ $p \mid \# [k]$. \square

A corollary: The two polynomials $x^p - x = x(x-1) \cdots (x-p+1)$ and $x^p - x$ are congruent modulo $p$ (where $p$ is prime), in the sense that corresponding coefficients are congruent (by Prop. 4.15 (b)). Note that their values at $x \in \mathbb{Z}$ are always $\equiv 0 \pmod{p}$, but they are not $\equiv 0 \pmod{p}$ as polynomials.
4.7. Eulerian numbers

Def. Let \( n, k \in \mathbb{N} \). Then, \( \binom{n}{k} \) denotes the number of \( \sigma \in S_n \) having exactly \( k \) descents.

(Recall: a descent of a \( \sigma \in S_n \) is an \( i \in [n-1] \) such that \( \sigma(i) > \sigma(i+1) \).)

These \( \binom{n}{k} \) are called Eulerian numbers.

Prop. 4.16. \( \binom{n}{k} = \binom{n-1-k}{k} \).

Proof. HW #4 exercise 3. \( \square \)

Prop. 4.17. (a) \( \binom{n}{0} = 1 \quad \forall n \in \mathbb{N} \).

(b) \( \binom{n}{k} = 0 \quad \forall k > n \).

(c) \( \binom{n}{0} = [k=0] \quad \forall k \in \mathbb{N} \).

(d) \( \binom{n}{k} = (k+1) \binom{n-1}{k} + (n-k) \binom{n-1}{k-1} \quad \forall \) positive integers \( n, k \).

Proof. (a)–(c) are easy. For (d), see MT #2. \( \square \)
Thm. 4.18. Let \( n \in \mathbb{N} \) and \( k \in \mathbb{N} \). Assume \( n > 0 \). Then,
\[
\binom{n}{k} = \sum_{i=0}^{k+1} (-1)^i \binom{n+1}{i} (k+1-i)^n.
\]

Proof (Stanley/Thomas, 2003): A \underline{k-barpe} (short for: \underline{k-barred permutation}) shall mean an \underline{n-tuple} containing each of the numbers \( 1, 2, \ldots, n \) exactly once (i.e., the one-line notation of some \( \sigma \in S_n \)), with (altogether) \( k \) bars placed between some of its entries (or at the very start, or at the very end), subdividing the \( n \)-tuple into \( k+1 \) (possibly empty) \underline{compartments}, with the property that the entries in each compartment are \underline{in increasing order}.

Examples (for \( n = 8 \)):
- 2 5-barpe: \( 5 \mid 1 \ 3 \mid 8 \ \| \ 2 \ 4 \ 6 \mid 7 \)
- 2 5-barpe: \( 5 \mid 1 \ 3 \ 8 \ \| \ 2 4 \ \| \ 6 \ 7 \)
- 2 5-barpe: \( \| 5 \mid 1 3 \ 8 \ \| 2 4 6 \ 7 \)
• not a $5$-barpe: $\frac{5}{3} \frac{1}{8} \frac{2}{4} \frac{6}{7}

\not\text{increasing} \quad \not\text{increasing}

Note:
• We omit commas & parentheses.
• Several bars can be placed between 2 consecutive entries (or at start or at end).
• We have
\[
(\text{# of } k\text{-barpes}) = (k+1)^n,
\]
(36)
since a $k$-barpe can be constructed by deciding, for each $i \in [n]$, which of the $k+1$ compartments we place $i$ in.

If $B$ is a $k$-barpe, then:
• a wall of $B$ means a bar of $B$ that is not immediately followed by another bar (e.g., $5|13|8|||246|7$).
• a useless wall of $B$ means a wall of $B$ such that removing that wall leaves a $(k-1)$-barpe (e.g., $5|13|8||246|7$).
Then, \( \langle \frac{n}{k} \rangle = (\# \text{ of } k\text{-barpes with no useless walls}) \)

(since \( k\text{-barpes with no useless walls are in bijection with permutations } \sigma \in S_n \text{ having } k \text{ descents,}

because each non-useless wall marks a descent).

Compute the RHS using Thm. 2.23 (PIE).

Let \( U = \{ \text{all } k\text{-barpes} \} \) (note: \( n \) is fixed).

For each \( i \in [n+1] \), let \( A_i \) be the set of all \( k\text{-barpes} \)

which have a useless wall between the \((i-1)\text{-st} \& i\text{-th}

entries (if \( i = 1 \), this means at front; if \( i = n+1 \), this

means at end). Then,

\[
\langle \frac{n}{k} \rangle = (\# \text{ of } k\text{-barpes with no useless walls})

= \left| U \setminus \bigcup_{i=1}^{n+1} A_i \right|
\]
Theorem 2.23(b) (applied to \( n+1 \) instead of \( n \))

\[
\sum_{I \subseteq [n+1]} (-1)^{|I|} \left| \bigcap_{i \in I} A_i \right|
\]

\[
= |\{ \text{\# of } (k-(|I|) \text{-barpes}) \text{ such that } \forall i \in I, \text{ the } k \text{-barpe } B \text{ has a useless wall between its } (i-1)\text{-st & } i\text{-th entries} \}| \]

\[
= |\{ (k-|I|) \text{-barpes} \}|
\]

by bijection: just remove the useless walls at those positions

\[
= \begin{cases} 
(k-|I|+1)^2 & \text{if } |I| \leq k \\
0 & \text{if } |I| > k 
\end{cases}
\]
\[
\begin{align*}
\sum_{I \subseteq \{n+1\} \atop |I| = i} (-1)^{|I|} & \begin{cases} 
(k-|I|+1)^n, & \text{if } |I| \leq k \\
0, & \text{if } |I| > k 
\end{cases} \\
= \sum_{i=0}^{n+1} \sum_{I \subseteq \{n+1\} \atop |I| = i} (-1)^i (k-i+1)^n, & \text{if } i \leq k \\
0, & \text{if } i > k \\
= \binom{n+1}{i} (-1)^i (k-i+1)^n, & \text{if } i \leq k \\
0, & \text{if } i > k
\end{align*}
\]
Exercise: Let \( n \in \mathbb{N} \).

For any \( f : [n] \rightarrow [n] \), let \( \text{Fix } f := \{ i \in [n] \mid f(i) = i \} \) be the set of fixed points of \( f \).

Find \( \sum_{\omega \in S_n} |\text{Fix } \omega| \).

Answer: \( n! \).

Proof:
\[
\sum_{\omega \in S_n} |\text{Fix } \omega| = \sum_{i \in [n]} \sum_{\omega \in S_n} [i \in \text{Fix } \omega]
\]

\[
= \sum_{i \in [n]} \sum_{\omega \in S_n} [i \in \text{Fix } \omega]
\]

\[
= \left( \text{# of } S_n \mid i \in \text{Fix } \omega \right)
\]

\[
= \left( \text{# of permutations of } [n] \setminus \{i\} \right)
\]

\[
= (n-1)!
\]

\[
= n!.
\]
Exercise: let $n \in \mathbb{N}$.

(2) For each $k \in \{0, 1, \ldots, n\}$, prove that $\sum_{\omega \in S_n} (|\text{Fix } \omega|) = \frac{n!}{k!}$.

(b) Find $\sum_{\omega \in S_n \text{ s.t. } \omega \text{ is even}} |\text{Fix } \omega|$.

4.9. The Lehmer code

Def. Let $n \in \mathbb{N}$.

(a) If $i \in [n]$ and $\omega \in S_n$, then $l_i(\omega) := (\# \ of \ j > i \ such \ that \ \omega(j) < \omega(i))$.

(b) If $\omega \in S_n$, then the $n$-tuple $(l_1(\omega), l_2(\omega), \ldots, l_n(\omega))$ is called the Lehmer code of $\omega$.

Def. Let $m \in \mathbb{Z}$. Then, $[m]_0 = \{0, 1, \ldots, m\}$.

Def. Let $(a_1, a_2, \ldots, a_n)$ and $(b_1, b_2, \ldots, b_n)$ be two $n$-tuples of integers. We say that $(a_1, a_2, \ldots, a_n) \preceq (b_1, b_2, \ldots, b_n)$ if & only if $\exists k \in [n]$ such that $a_k < b_k$, but $a_i = b_i \ \forall i < k$. 

Examples: \[ (3, 4, 6) \preceq_{\text{lex}} (3, 5, 2), \]
\[ (4, 1, 2, 5) \preceq_{\text{lex}} (4, 1, 3, 0), \]
\[ (1, 1, 0, 1) \preceq_{\text{lex}} (2, 1, 0, 0). \]

Thm. 4.19. Let \( n \in \mathbb{N} \).

(a) For each \( \sigma \in S_n \), we have \( l(\sigma) = l_1(\sigma) + l_2(\sigma) + \ldots + l_n(\sigma). \)

(Note: \( l_n(\sigma) = 0 \).

(b) If \( \sigma, \tau \in S_n \) satisfy
\[ (\sigma(1), \sigma(2), \ldots, \sigma(n)) \preceq_{\text{lex}} (\tau(1), \tau(2), \ldots, \tau(n)), \]
then \( (l_1(\sigma), l_2(\sigma), \ldots, l_n(\sigma)) \preceq_{\text{lex}} (l_1(\tau), l_2(\tau), \ldots, l_n(\tau)). \)

(c) The map
\[ L : S_n \rightarrow [n-1]_0 \times [n-2]_0 \times \ldots \times [n-n]_0 \]
\[ \sigma \mapsto (l_1(\sigma), l_2(\sigma), \ldots, l_n(\sigma)) \]
is well-defined & bijective.
Proof of Proposition 4.9.

\[ \sum_{\omega \in S_n} x^L(\omega) = \sum_{\sigma \in S_n} x^L(\sigma) \]

\[ \overset{\text{Thm. 4.19(a)}}{=} \sum_{\sigma \in S_n} x^{l_1(\sigma)+l_2(\sigma)+\ldots+l_n(\sigma)} \]

\[ \overset{\text{Thm. 4.19(c)}}{=} \sum_{(i_1,i_2,\ldots,i_n) \in [n-1]_0 \times [n-2]_0 \times \ldots \times [n-n]_0} x^{i_1+i_2+\ldots+i_n} = x^{i_1} x^{i_2} \ldots x^{i_n} \]

\[ = \left( \sum_{i_1 \in [n-1]_0} x^{i_1} \right) \left( \sum_{i_2 \in [n-2]_0} x^{i_2} \right) \ldots \left( \sum_{i_n \in [n-n]_0} x^{i_n} \right) \]

\[ = (1 + x + x^2 + \ldots + x^{n-1}) (1 + x + x^2 + \ldots + x^{n-2}) \ldots (1) \]

\[ = \prod_{i=1}^{n-1} (1 + x + x^2 + \ldots + x^i) = \prod_{i=1}^{n-1} (1 + x + x^2 + \ldots + x^i). \]
Thm. 4.20. Let $n \in \mathbb{N}$, let $\sigma \in S_n$.

For each $i \in \lbrack n \rbrack$, let $a_i = \text{cyc } i, i^{-1}, \ldots, i = s_{i-1} s_{i-2} \cdots s_i$,

where $i' = i + \ell_i(\sigma)$.

Then, $\sigma = a_1 a_2 \cdots a_n$. (Note: $a_n = \text{id}$.)

Proof: MT#2?

Note that Thm. 4.20 yields a new, explicit proof of Thm. 4.6.

Remark. Let $n \in \mathbb{N}$ and $\sigma \in S_n$. Here is a visual way to think of the Lehmer code:

- Draw an (empty) $n \times n$-matrix.
- For each $i \in \lbrack n \rbrack$, put an $\times$ into its cell $a(i, \sigma(i))$.

Running example: $n = 6$ and $\sigma = [5, 4, 3, 4, 6, 2]$ (in 1-line notation).
From each X, draw a vertical line downwards & a horizontal line eastwards.

Draw 2 o into each cell that does not fall on any line.

This is called the Rothe diagram of $\sigma$. 
Explicitly: a cell \((i,j)\) has a \(0\) in it

\[\iff \sigma(i) = j \land \sigma^{-1}(j) > i.\]

In other words, a cell \((i, \sigma(j))\) has a \(0\) in it

\[\iff \sigma(i) = \sigma(j) \land \sigma^{-1}(j) > i,\]

\[\iff (i,j) \text{ is an inversion of } \sigma.\]

Thus, \(l(\sigma) = (\text{# of inversions of } \sigma) = (\text{# of 0's}).\)

Furthermore, \(l_i(\sigma) = (\text{# of 0's in row } i) \quad \forall i \in [n].\)

Finally, let us label the 0's as follows:

for each \(i \in [n]\), label the 0's in row \(i\) from right to left by \(s_i, s_{i+1}, s_{i+2}, \ldots, s_{i-1}\)

where \(i' = i + l_i(\sigma)\).

Then, read the matrix row by row, starting with the top row, from left to right. \(\implies\) This gets you a product of simples that equals \(\sigma\) (by Thm. 4.20).
4.10. Permutation patterns (a glimpse)

Def. Let \( m, n \in \mathbb{N} \), let \( \tau \in S_m \) and \( \sigma \in S_n \).

A \( \tau \)-pattern in \( \sigma \) means a subsequence of \( (\sigma(1), \ldots, \sigma(n)) \) whose entries come in the same order as \( \tau(1), \ldots, \tau(m) \).

(Rigorously: it means an \( m \)-tuple \( (i_1 < i_2 < \ldots < i_m) \) of numbers in \( [n] \) such that \( \forall x, y \in [m], \) we have
\[
(\sigma(i_x) < \sigma(i_y) \iff \tau(x) < \tau(y)),
\]

We say \( \sigma \) is \( \tau \)-avoiding if \( \not\exists \) \( \tau \)-pattern in \( \sigma \).

Example: A \( 21 \)-pattern in \( 2 \) \( \sigma \in S_n \) is an inversion of \( \sigma \).

\( \sigma = [2, 1] \)

Thus, the only \( 21 \)-avoiding \( \sigma \in S_n \) is \( \text{id} \).

- A \( 12 \)-pattern in \( 2 \) \( \sigma \in S_n \) is a non-inversion (= a pair \( (i, j) \) with \( 1 \leq i < j \leq n \) and \( \sigma(i) < \sigma(j) \)) of \( \sigma \).
A $123$-pattern in a $\sigma \in S_n$ is a triple $(i<j<k)$ with $\sigma(i) < \sigma(j) < \sigma(k)$.

A $231$-pattern in a $\sigma \in S_n$ is a triple $(i<j<k)$ with $\sigma(k) < \sigma(i) < \sigma(j)$.

  - 132-avoiding? 21534, so no.
  - 231-avoiding? yes!
  - 321-avoiding? 21534, so yes.

Thm. 4.21. Let $n \in \mathbb{N}$. Then,

\[
\text{(\# of 123-avoiding perms $\sigma \in S_n$)}
\]

\[= \text{(\# of 132- )}
\]

\[= \text{(\# of 213- )}
\]

\[= \text{(\# of 231- )}
\]

\[= \text{(\# of 312- )}
\]
\[
= \left( \# \text{ of } 321- \right) \\
= \frac{1}{n+1} \binom{2n}{n} = (\frac{2n}{n}) - (\frac{2n}{n-1})
\]

(a so-called Catalan number).

(For more about patterns, see:
* [Kitaev: Patterns in Permutations and Words],
* [Bóna: Combinatorics of Permutations].)