Continuing the proof of Thm. 8.6 (the "O" was premature):

\[\Rightarrow: \text{Assume } [x^0]a \text{ has an inverse in } K.\]

Write \( a \) as \((a_0, a_1, a_2, \ldots)\), and try to find an FPS \( b = (b_0, b_1, b_2, \ldots) \) with \( ab = 1 \).

So we want

\[
\begin{align*}
(1, 0, 0, 0, \ldots) &= 1 = ab = (a_0, a_1, a_2, \ldots) (b_0, b_1, b_2, \ldots) \\
&= (a_0b_0, a_0b_1 + a_1b_0, a_0b_2 + a_1b_1 + a_2b_0, \ldots).
\end{align*}
\]

So we want

\[
\begin{align*}
1 &= a_0b_0, \\
0 &= a_0b_1 + a_1b_0, \\
0 &= a_0b_2 + a_1b_1 + a_2b_0, \\
0 &= a_0b_3 + a_1b_2 + a_2b_1 + a_3b_0, \\
&\quad \quad \quad \quad \quad \quad \vdots
\end{align*}
\]
Theorem 8.7. \( \frac{1}{1-x} \) has an inverse, which is \( 1 + x + x^2 + x^3 + \cdots \). Hence, \( \frac{1}{1-x} \) is the FPS for each \( n \in \mathbb{N} \), since a \( k \)-make \( a_1 \) has a (mult.) inverse \( a^{-1} \), then we can define \( a^{-n} \) for each \( n \in \mathbb{N} \).

Proof. If \( a \in K[x] \) has a (mult.) inverse \( a^{-1} \), then we can define \( a^{-n} \) for each \( n \in \mathbb{N} \), since \( a_0 = \frac{1}{a_1} \) has a mult. inverse and thus can be divided by.

Solve this system by elimination: get \( \alpha_1 \) from the next, etc.

this can be done, since \( a_0 = \frac{1}{a_1} \) has a mult. inverse.
2nd proof:

\[(1-x)(1+x+x^2+x^3+\ldots) = (1-x^2)+(x-x^2)+\left(x^2-x^3\right)+\left(x^3-x^4\right)+\ldots = 1. \]

\[\text{Thm. 8.8 ("Newton's binomial theorem").}\]

\[(1+x)^n = \sum \binom{n}{k} x^k \quad \forall n \in \mathbb{Z},\]

\[\text{actually an \infty sum if } n < 0\]

\[\text{Proof idea: For } n > 0, \text{ this follows from the regular binomial thm.}\]

\[\text{For } n = -1, \text{ this says } (1+x)^{-1} = \sum \frac{(-1)^k}{k} x^k, \text{ which is similar to Thm. 8.7.}\]

\[\text{Lem. 8.9. } (1+x)^{-n} = \sum \binom{-n}{k} x^k \quad \forall n \in \mathbb{N}.\]

\[\text{Proof idea for Lem. 8.9. Induction on } n.\]

\[\text{For the Ind. step, we need to check:}\]
\[
\left( \sum_{k} (-1)^{k} \binom{n+k-1}{k} x^k \right) \cdot (1+x)^{-1} = \sum_{k} (-1)^{k} \binom{n+k}{k} x^k.
\]

Equivalently,
\[
\sum_{k} (-1)^{k} \binom{n+k-1}{k} x^k = \left( \sum_{k} (-1)^{k} \binom{n+k}{k} x^k \right) \cdot (1+x)^{-1}.
\]

The RHS can be rewritten as
\[
\sum_{k} (-1)^{k} \binom{n+k}{k} x^k + \sum_{k} (-1)^{k} \binom{n+k}{k} x^{k+1}
\]
\[
= \sum_{k} (-1)^{k-1} \binom{n+k-2}{k-1} x^k
\]
\[
= \sum_{k} \left[ (-1)^{k} \binom{n+k}{k} + (-1)^{k-1} \binom{n+k-1}{k-1} \right] x^k
\]
\[
= (-1)^{k} \left( \binom{n+k}{k} - \binom{n+k-1}{k-1} \right)
\]
\[
= (-1)^{k} \binom{n+k-1}{k} \text{ (by the recurrence of binomial coefficients)}
\]
\[= \sum_{k=0}^n (-1)^k \binom{n+k-1}{k} x^k,\]

which is the LHS.

8.5. Substitution

Prop. Let \( f \) and \( g \) be two FPS with \( [x^0]g = 0 \) (that is, \( g = g_1 x + g_2 x^2 + g_3 x^3 + \ldots \)).

Then, the FPS \( f \circ g \) (also known as \( f(g) \)) is defined as follows:

Write \( f \) as \( f = \sum_{n \geq 0} f_n x^n \) and set \( f \circ g = \sum_{n \geq 0} f_n g^n \).

We call \( f \circ g \) the composition of \( f \) with \( g \), or the result of substituting \( g \) for \( x \) in \( f \).

We will NOT call it \( f(g) \), to avoid clashing with product notation.
The sum \( \sum_{n \geq 0} f_n g^n \) in the above definition is well-defined, i.e., the family \( (f_n g^n)_{n \in \mathbb{N}} \) is summable, since

\[ (86) \]

the first \( n \) coefficients of \( g^n \) are 0, \( \forall n \in \mathbb{N}. \)

(86) is easy to prove by induction on \( n. \)

Alternatively: Write \( \mathbb{N} \) \( g = xh \) for some FPS \( h, \) since

\[ [\mathbb{N} \text{ x}^n g = 0. \) Thus, \( g^n = x^n h^n. \)

Example: We can substitute \( x + x^2 \) for \( x \) into \( 1 + x + x^2 + \ldots. \)

The result is

\[ 1 + (x + x^2) + (x + x^2)^2 + (x + x^2)^3 + (x + x^2)^4 + \ldots. \]

\[ = 1 + x + 2x^2 + 3x^3 + 5x^4 + 8x^5 + \ldots. \]

\[ = \sum_{n \geq 0} f_{n+1} x^n, \]

where \( f_i = \) Fibonacci \( \# \) s.

This is because substituting \( x + x^2 \) for \( x \) in
\[ 1 + x + x^2 + \ldots = \frac{1}{1-x} \]

yields
\[ 1 + (x+x^2) + (x^2+x^2)^2 + (x+x^2)^3 + \ldots \]
\[ = \frac{1}{1-(x+x^2)} = \frac{1}{1-x-x^2} \]
\[ \text{Ex. 1.} \quad 88.1. \quad \sum_{n=0}^{\infty} f_{n+1} x^n, \]

Here, we have tacitly used:

Prop 8.10. Substitution satisfies the rules you would expect:
\[ (g_1+g_2) \circ h = g_1 \circ h + g_2 \circ h \]
\[ (g_1g_2) \circ h = (g_1 \circ h)(g_2 \circ h) \]
\[ f \circ (g \circ h) = (f \circ g) \circ h \]

(See [Hoehn, Ch. 7] for details.)

This all justifies Ex. #1 in 88.1.

Rmk. A polynomial is a FPS \((c_0, a_1, a_2, \ldots)\) such that all but finitely many \( i \in \mathbb{N} \) satisfy \( a_i = 0 \).
To justify Ex. 2, we need to define \((1+x)^n\) for \(n \in \mathbb{Z}\).

**Option 1:** define \((1+x)^n = \sum_{k} \binom{n}{k} x^k\).

But then, we would have to prove all the rules of exponents:
\[
(1+x)^n(1+x)^m = (1+x)^{n+m},
\]
\[
((1+x)^n)^m = (1+x)^{nm},
\]
\[\text{etc.}\]

**Option 2:** define \((1+x)^n = \exp \left(n \log (1+x)\right)\).

What are \(\exp\) and \(\log\)? We define
\[
\exp = \sum_{n \geq 0} \frac{1}{n!} x^n, \quad \log (1+x) = \sum_{n \geq 1} \frac{(-1)^{n-1}}{n} x^n.
\]

So \((1+x)^n = \exp \circ \left(n \log (1+x)\right)\).

You would still have to prove many things, but this is more doable. See [Loehr] or [my log/exp notes from Fall 2017 Math 4930J].
8.6. Another example

Here is another application of FPS; see [Galvin, §40 (1)].

Def. Let \( n \in \mathbb{N} \), let \( \sigma \in S_n \). The order of \( \sigma \) is the smallest \( k > 0 \) such that \( \sigma^k = id \).

Prop. 8.11. Let \( n \in \mathbb{N} \) and \( \sigma \in S_n \). The order of \( \sigma \) is well-defined & equals the lcm (= least common multiple) of the lengths of the cycles of \( \sigma \).

Proof idea. You just need to show that an \( m \in \mathbb{N} \) satisfies \( \sigma^m = id \) if & only if the length of each cycle of \( \sigma \) divides \( m \).

For each \( n \in \mathbb{N} \), let \( a_n \) be the number of permutations \( \sigma \in S_n \) having odd order. What is \( a_n \)?
Ex: \( n = 4: \)

\[
\begin{array}{|c|c|}
\hline
\text{id} \quad \text{order 1} & \text{8 permutations} \quad \text{order 3} \\
\hline
\end{array}
\]

\[ \Rightarrow a_4 = 9 \]

Similarly, \( a_3 = 3 \) and \( a_2 = 1 \) and \( a_5 = 45. \)

(OEIS: A000246.)

First observation: A perm. \( \sigma \in S_n \) has odd order \( \iff \) all cycles of \( \sigma \) have odd lengths. Thus,

\[
a_n = \sum_{(i_1, i_2, \ldots, i_n) \in \mathbb{N}^n} \quad \text{(# of } \sigma \in S_n \text{ with } i_j \text{ cycles of length } j \text{)}
\]

\[
\begin{aligned}
&i_k = 0 \quad \forall \text{ even } k; \\
i_k > 0 \quad \forall \text{ odd } k; \\
i_1 + 2i_2 + \cdots + ni_n = n
\end{aligned}
\]

\[
= \frac{n!}{\prod i_k! \cdot i_1! \cdot i_2! \cdots i_n! \cdot 1^2 \cdot 2^2 \cdots n^2}
\]

(by Exercise after Prop. 4.13)
\[ \sum \frac{n!}{i_1! i_2! \cdots i_n! 1^{i_1} 2^{i_2} \cdots n^{i_n}} \]

Divide this multiply this by \( \alpha^x \)

\[ \sum \frac{n!}{i_1! i_2! \cdots 1^{i_1} 2^{i_2} \cdots} \]

\[ \text{Multiply this identity by } \frac{x^n}{n!} \text{ to get} \]
\[
\frac{a_n x^n}{n!} = \sum_{(i_1, i_2, i_3, \ldots) \in \mathbb{N}^\infty} \sum_{i_k = 0}^{n} \ldots \prod_{k} \frac{1}{i_k!} \left( \frac{x^{i_1} x^{2i_2} \cdots}{x^{i_1} x^{2i_2} \cdots} \right)
\]

Summing this over all \( n \in \mathbb{N} \), we get

\[
\sum_{n \in \mathbb{N}} \frac{a_n x^n}{n!} = \sum_{(i_1, i_2, i_3, \ldots) \in \mathbb{N}^\infty} \sum_{i_k = 0}^{n} \ldots \prod_{k} \frac{1}{i_k!} \left( \frac{x^{i_1} x^{2i_2} \cdots}{x^{i_1} x^{2i_2} \cdots} \right)
\]
\[
\sum_{(i_1, i_2, i_3, \ldots) \in \mathbb{N}^\infty; i_k = 0 \text{ for all but finitely many } k} \frac{x^{1i_1} x^{3i_3} \ldots}{i_1! i_3! \ldots i_k^{i_k}} = \prod_{k \geq 1 \text{ odd}} \left( \sum_{i \in \mathbb{N}} \frac{x^{ki}}{i! k^i} \right)
\]

(Note that infinite products of PDFs can make sense just as infinite sums do.)

\[
= \prod_{k \geq 1 \text{ odd}} \left( \sum_{i \in \mathbb{N}} \frac{x^{ki}}{i! k^i} \right) = \prod_{k \geq 1 \text{ odd}} \exp \left( \frac{x^k}{k} \right)
\]
\[ = \exp \left( \sum_{k \geq 1 \text{ odd}} x^k/k \right) \]

(here we used the rule
\[ \prod \exp(x_i) = \exp \left( \sum_{i \in I} x_i \right), \]
\[ \text{which is not hard to check}. \]

But \[ \sum_{k \geq 1 \text{ odd}} x^k/k \]
\[ = \frac{1}{2} \left( \sum_{k \geq 1} x^k/k - \sum_{k \geq 1} (-x)^k/k \right) \]
\[ \overset{\text{"destructive interference"}}{=} -\log(1-x) - (-\log(1+x)) \]
\[ = \frac{1}{2} \left( -\log(1-x) - (-\log(1+x)) \right) \]
\[ = \frac{1}{2} \left( \log(1+x) - \log(1-x) \right) \]

so this becomes
\[ \sum_{n \in \mathbb{N}} \frac{a_n x^n}{n!} = \exp \left( \frac{1}{2} \left( \log (1+x) - \log (1-x) \right) \right) \]

\[ = \left( \frac{1+x}{1-x} \right)^{1/2} \]

\[ = \frac{((1+x)(1-x))^{1/2}}{1-x} = \frac{(1-x^2)^{1/2}}{1-x} \]

\[ = (1-x^2)^{1/2} \cdot \frac{(1-x)^{-1}}{1-x} = \sum_{k \geq 0} x^k \]

\[ = \left( \sum_{k \geq 0} \frac{1}{k!} (-x^2)^k \right) \left( \sum_{k \geq 0} x^k \right) \]

\[ = \left( \sum_{k \geq 0} \frac{1}{k!} (-1)^k x^{2k} \right) \left( \sum_{k' \geq 0} x^{k'} \right) \]
\[ = \sum_{n>0} \left( \sum_{k\leq n/2} \binom{n/2}{k} (-1)^k \hat{x}^k \right) x^n. \]

Comparing coefficients, we get

\[ \frac{a_n}{n!} = \sum_{k\leq n/2} \binom{n/2}{k} (-1)^k = \sum_{k=0}^{\ln/2}\binom{-1/2}{k} (\frac{1/2}{\ln/2})^k \]

\[ = (-1)^{\ln/2} \binom{1/2-1}{\ln/2} \]

(by HW #2 exercise 4, applied to \( \ln/2 \))
and 1/2 instead of \( m \) and \( n \)
(since we did not need \( n \in N \) in that exercise)

\[ = (-1)^{\ln/2} \left( -\frac{1/2}{\ln/2} \right) \frac{\text{HW#3 ex3(a)}}{(-1)^{\ln/2} \left( -\frac{1/4}{\ln/2} \right) \left( \frac{2\ln/2}{\ln/2} \right)} \]

\[ = (1/4)^{\ln/2} \]
\[ \left( \frac{1}{4} \right)^{\ln(2)} \left( \frac{2\ln(2)}{\ln(2)} \right) \]

Thus,

\[ a_n = n! \cdot \left( \frac{1}{4} \right)^{\ln(2)} \left( \frac{2\ln(2)}{\ln(2)} \right) \]

9. **Partitions**

9.4. **Basics:**

Recall (from §3.6): A **partition** of an \( n \in \mathbb{Z} \) means a weakly decreasing tuple of positive integers with sum \( n \).

The entries of a partition are called its **parts**.

\[ p(n) : = (\# \text{ of partitions of } n) \]

\[ p_k(n) : = (\# \text{ of partitions of } n \text{ into } k \text{ parts}) \]
Prop. 3.13 (e) yields
\[ p_k(n) = p_k(n-k) + p_{k-1}(n-k) \]
\[ \forall k \geq 1 \quad \forall n \in \mathbb{N}. \]

Also,
\[ p(n) = p_0(n) + p_1(n) + \ldots + p_n(n). \]

Thm. 9.1.
\[ \sum_{n \geq 0} p(n)x^n = \prod_{k=1}^{\infty} \frac{1}{1-x^k}. \]

(The product on the RHS is well-defined, because multiplying a FPS by \( \frac{1}{1-x^k} \) does not affect its first \( k \) coefficients.)

Proof.
\[ \prod_{k=1}^{\infty} \frac{1}{1-x^k} = \prod_{k=1}^{\infty} (1 + x^k + (x^k)^2 + (x^k)^3 + \ldots) \]
\[ = \prod_{k=1}^{\infty} (1 + x^k + x^{2k} + x^{3k} + \ldots) \]
\[= \left(1 + x + x^2 + x^3 + x^4 + \ldots \right)\]
\[\cdot \left(1 + x^2 + x^4 + x^6 + x^8 + \ldots \right)\]
\[= \left(1 + x^3 + x^6 + x^9 + \ldots \right)\]
\[= \left(1 + x^4 + x^8 + x^{12} + \ldots \right)\]
\[= \left(1 + x^5 + x^{10} + x^{15} + \ldots \right)\]
\[= 1 + x + 2x^2 + 3x^3 + 5x^4 + \ldots\]

Ways to get \(x^4\):

\[
\begin{array}{|c|c|c|c|c|}
\hline
(0, 0, 0, 0) & (4, 0, 0) & (2, 1, 0) & (1, 0, 1) & (0, 0, 1) \\
\hline
\end{array}
\]
What is the coefficient of $x^n$ for a general $n \in \mathbb{N}$?

It is the number of ways to assemble $x^n$ by picking an addend out of each factor.

In other words: It is the number of all $(m_1, m_2, m_3, \ldots) \in \mathbb{N}^\infty$ such that $1m_1 + 2m_2 + 3m_3 + \ldots = n$.

But

$$\{\text{partitions of } n\} \rightarrow \{ (m_1, m_2, m_3, \ldots) \in \mathbb{N}^\infty \mid 1m_1 + 2m_2 + 3m_3 + \ldots = n \}$$

$$\lambda \mapsto (\text{# of parts } 1 \text{ in } \lambda, \text{# of parts } 2 \text{ in } \lambda, \text{# of parts } 3 \text{ in } \lambda, \ldots)$$

is a bijection. Thus, our coefficient is the number of partitions of $n$. But this is $p(n)$. \[\square\]